CHAPTER 7

CHAOTIC CONTACT BIFURCATIONS

Chaotic contact bifurcations involve a chaotic attractor. This is the pinnacle of our subject. Here we proceed with a 1D introduction, and a 2D introduction, before analyzing the exemplary bifurcation sequence.

7.1 BIFURCATIONS IN ONE DIMENSION

Recall from Chapter 3 that the basic critical curve L (as defined in 3.2) belongs to the frontier of zones having different numbers of preimages, for example, between Z_1 and Z_3. These concepts were introduced in the one-dimensional case in 2.2.

We now turn to another context, the *chaotic contact bifurcation*, or *CCB*. Critical points are fundamental to an understanding of these global events, and this understanding will be very useful when we come to consider CCBs in the two-dimensional case.

In one-dimensional iterations, the transition to chaos (a bifurcation sequence in which a chaotic attractor is created out of the blue sky, or a periodic attractor becomes chaotic) has been a primary concern since the pioneering works of Myrberg in 1958 (see Appendices 5 and 6.) The role of the critical points in these transitions has been explored by Mira since 1975. His analysis of the box-within-a-box bifurcation structure is described in M1. The connection between bifurcations due to the critical points and the homoclinic bifurcations of the repelling cycles has been presented in (Gardini, 1994). (Homoclinic points were discussed in Chapter 5.)

This latter is the subject of this chapter. We are going to illustrate two kinds of CCB. These examples are also homoclinic bifurcations. Both concern a chaotic attractor having several pieces — intervals in the one-dimensional case — which are permuted cyclically by the map. These are called *cyclical chaotic attractors*.

Warning: We use the term *chaotic attractor*, loosely, for a situation revealed experimentally. We can never be sure that a trajectory which appears chaotic is a true chaotic attractor, or just a very long chaotic transient.

In a *CCB of the first kind* a cyclical chaotic attractor explodes into a single, larger chaotic attractor. In a *CCB of the second kind*, a $2k$-cyclic chaotic attractor is transformed into a k-cyclic chaotic attractor as pieces merge pairwise.

Our examples will all occur in the quadratic family of maps of the real line studied by Myrberg, $f(x) = x^2 - b$. We will discuss the dynamics as b increases in the interval $[-2, 2]$. We begin with $b = 1.0$ and increase to about $b = 1.8$, in nine stages.

The graph of a Myrberg map is a parabola in standard position, except for the vertical displacement by $-b$. As the bifurcation parameter b increases, the parabola descends. As described in Chapter 2, two fixed points appear in a fold bifurcation as b increases through the value 0.25. Generally, a fold bifurcation is the opening of a *box of the first kind*, in the language of Mira. A *box* is an interval in the one-dimensional space of the parameter, b.

Stage 1: $b = 1.0$

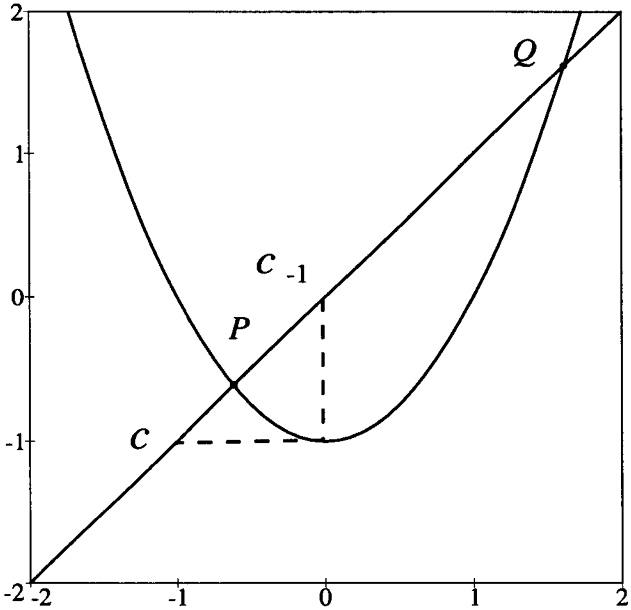
Figure 7-1 shows the main qualitative features of the function f. The two fixed points are shown by the intersection of the graph of f and the diagonal line. Although these points are really located in the domain of the map (the horizontal axis in this figure), we show them above the horizontal axis, on the graph of the function or on the diagonal, which are the same in this case. This convention is very useful, and will be followed throughout this chapter.

The critical point (value of x at which $f(x)$ achieves its minimum) is the origin, 0, and is denoted c_{-1} here. The critical value $f(c_{-1})$ is indicated by c. Of course, this is just the critical value $-b$. Again, both are shown on the diagonal, following our convention.

The fixed point Q is repelling, while the fixed point P, although initially attracting when created at $b = 0.25$, underwent a period-doubling (flip) bifurcation when b increased past approximately 0.7495, as described in Chapter 2. So at the current value, 1.0, P is a repellor. This flip is the first in the Myrberg sequence, well studied in the works of Feigenbaum. In the box-within-a-box language of Mira, this flip is the opening of the first box of the second kind, an small interval of the b space, within a box of the first kind, a larger interval. Each box of the first kind includes a related box of the second kind.

FIGURE 7-1.

$b = 1.0$

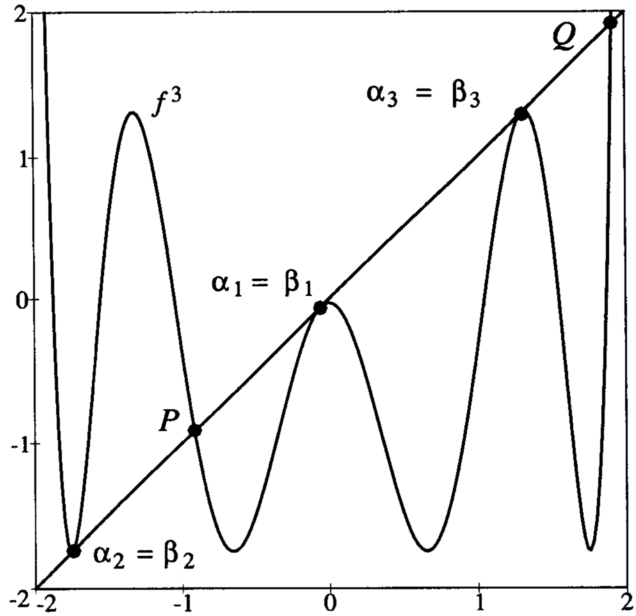


Stage 2: $b = 1.75$

A k-periodic point of the map f is a fixed point of the map f^k (f iterated k times). For example, to find a 3-cycle of f, we look for three related intersections of f^3 with the diagonal. In Figure 7-2, we see a fold bifurcation of f^3 occurring. The fixed points, P, Q, of f are still fixed points of f^3, so they appear here as intersections, as marked. But there are three new contacts as well, labelled $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, and $\alpha_3 = \beta_3$. These comprise a 3-cycle of f, as well as fixed points of f^3. A box of the first kind opens here.

FIGURE 7-2.

$b = 1.75$



Stage 3: $b = 1.76$

Just after the fold bifurcation, there are two 3-cycles, one $\{\alpha_1, \alpha_2, \alpha_3\}$ attracting, the other $\{\beta_1, \beta_2, \beta_3\}$ repelling, as described in Chapter 2 and shown in Figure 7-3.

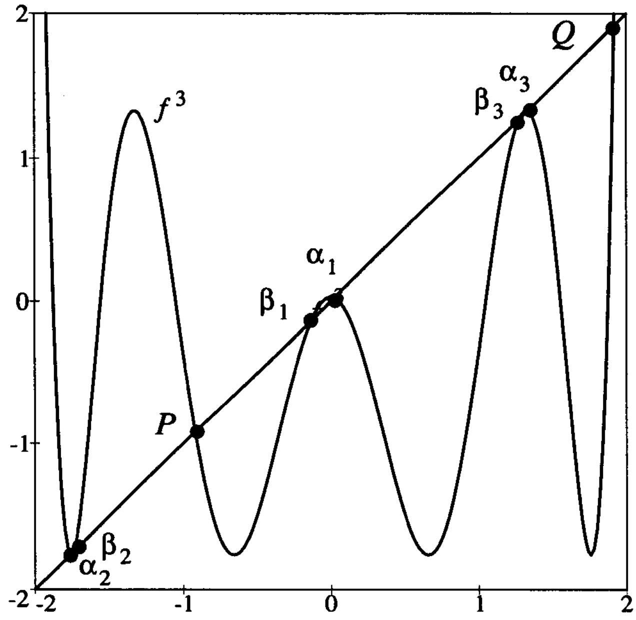


FIGURE 7-3.
$b = 1.76$

Stage 4: $b = 1.771$

At this value of b, an attracting 6-cycle has appeared recently, when the formerly attracting 3-cycle of the preceding stage flipped and became a repelling 3-cycle, opening the related box of the second kind. Figure 7-4 shows the attractive 6-cycle, both on the graph of f and on the diagonal.

Figure 7-5 shows a small rectangular neighborhood of the repelling 3-periodic point α_3, with two 6-periodic points nearby, both attracting. They are all shown on the crossing of the diagonal with the graph of f^6, as they are fixed points of this function.

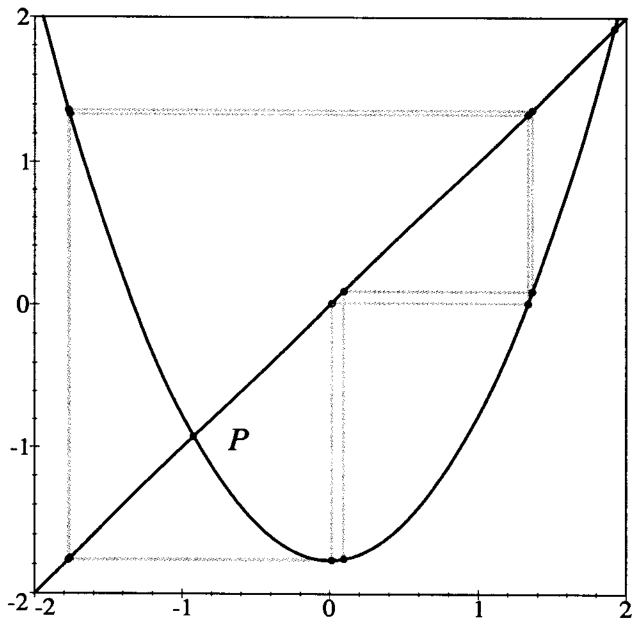


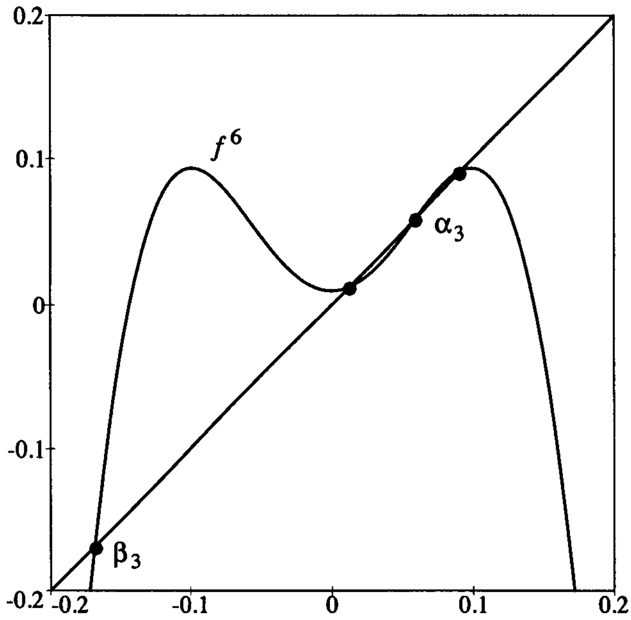
FIGURE 7-4.

$b = 1.771$

The six points of the 6-cycle (two points at the left are very close) and the fixed point, P, shown on the diagonal.

FIGURE 7-5.

$b = 1.771$



Stage 5: $b = 1.781$

Figure 7-6 shows that as b increases a bit more, the orbit shown seems to be chaotic in six disjoint intervals, rather than cyclic. Figure 7-7 is an enlargement of the pair of chaotic intervals in the center of Figure 7-6, which nearly abut the 3-periodic repellor, α_3. This enlargement shows evidence that a attractor fills the six intervals, and thus is chaotic (not cyclic).

FIGURE 7-6.

$b = 1.781$

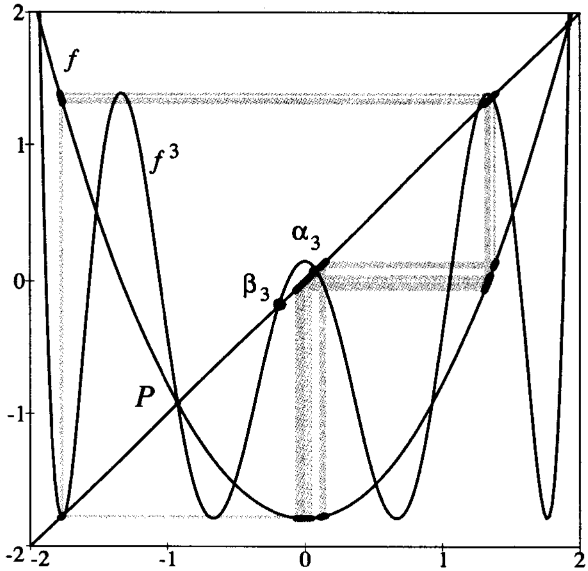
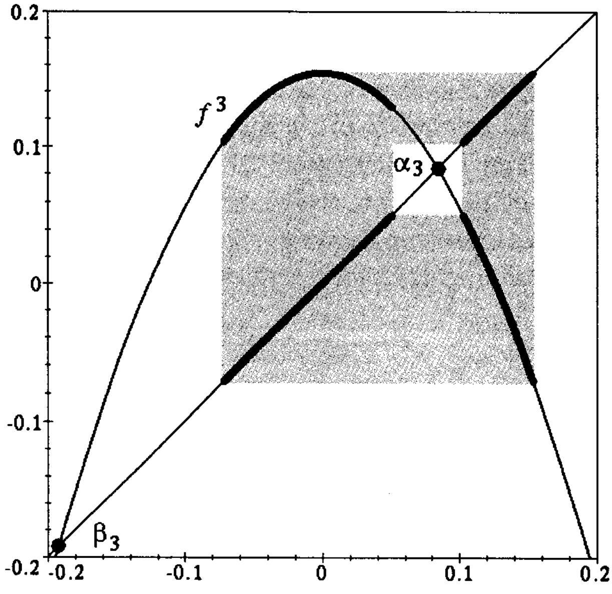


FIGURE 7-7.

$b = 1.781$



Stage 6: $b = 1.7822$

This is the moment of contact bifurcation. A critical point has moved toward, and now makes contact with, the point α_3 of the repelling 3-cycle $\{\alpha_1, \alpha_2, \alpha_3\}$. In this stage we see the closure of the box of the second kind which opened with the flip bifurcation of the α cycle, discussed in Stage 4.

Figure 7-8 shows a blowup of the same piece of f^3 seen in Figure 7-7. Here we see a trajectory of the basic critical point c_{-1} through three iterations of f^3 (that is, c_{-1}, c_2, c_5, c_8) until c_8 falls on α_3. Also, four preimages of c_{-1} are shown, indicating that a backward sequence of critical points, $c_{-4}, c_{-7}, c_{-10}, \ldots$, approaches α_3 asymptotically. The trajectory is shown in the Koenig-Lemeray style explained in Chapter 2.

With this contact there appear for the first time orbits homoclinic to α_3. These are points with orbits which are initially repelled by, but later jump back onto, α_3. A point like α_3 that has a homoclinic trajectory is called a *snap-back repellor* (SBR). In this figure we show three points c_{-1}, c_2, and c_5, which are homoclinic to α_3. They lie on opposite sides of the SBR α_3.

Before this bifurcation, the attractor occupied six chaotic intervals, permuted cyclically by the map f (and this is therefore called a 6-cyclic chaotic attractor). After this bifurcation, as we will see in the next figures, the attractor occupies only three (larger) chaotic intervals, permuted cyclically by the map f (a 3-cyclical chaotic attractor). Thus, we have at this instant an example of a CCB of the second kind, as described at the beginning of this chapter. The six intervals have joined pairwise into three larger intervals.

We may also see, in Figure 7-8, an example of an absorbing interval. This is an interval bounded by critical points, which is mapped into itself and which is absorbing in the sense that all points sufficiently near will jump into the interval in a finite number of iterations. In this figure, the interval $[c_5, c_2]$ is absorbing. It contains a chaotic attractor of f^3. Also, it is invariant under the map f^3, that is, it is mapped exactly onto itself. On the other hand, under the map f, the three intervals — $[c_5, c_2]$ and its two images under the map, f — comprise an absorbing set which contains the 3-cyclic chaotic attractor. We also call these three component intervals of the absorbing set, collectively, *absorbing intervals*.

Note: The intervals are not individually mapped into themselves by f, but only by f^3. Their union is mapped into itself by f. Note also that before this homoclinic bifurcation, which is a CCB of the second kind, critical points define the boundary of 2·3-cyclic[1] absorbing intervals, which include the 2·3-cyclic attractor. The points α_i of the repelling 2·3-cycle, together with their preimages of all ranks, define the boundary of the *basin of attraction* of the 2·3-cyclic absorbing intervals, and each α_i separates two *immediate basins*. These concepts have been introduced in Chapter 2. When this CCB of the repelling 3-cycle occurs, the map has 6-cyclic chaotic intervals, not distinct, as well as 3-cyclic chaotic intervals. That is, the map f^6 has six invariant chaotic intervals, not disjoint, and the map f^3 has three disjoint invariant chaotic intervals.

The situation is similar for any CB of the second kind, in which a $2k$-cyclic chaotic attractor is transformed into a k-cyclic chaotic attractor: the *closure* of a box of the second kind is characterized by the appearance (for the first time) of homoclinic orbits of the k-cycle, α, on both sides of the points α_i of the cycle, and the $2k$ absorbing intervals merge into k absorbing intervals. This occurs without an abrupt increase in the size of the absorbing set, such as we will see in the next example.

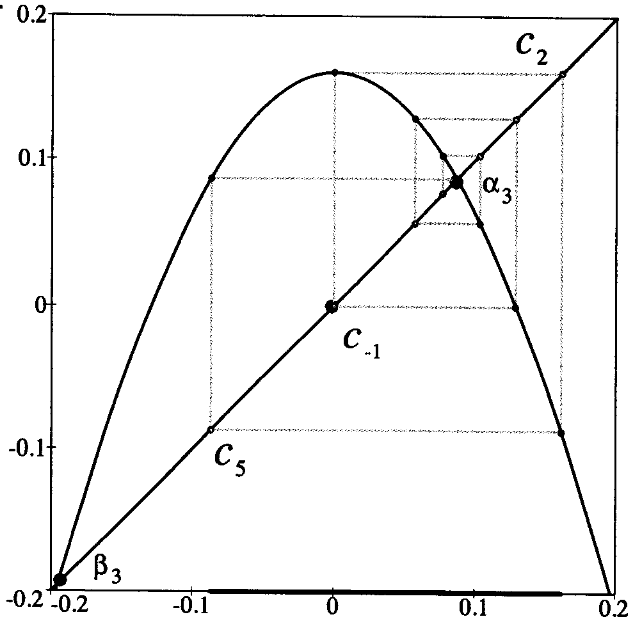
So far we have introduced most of the main ideas pertaining to CCBs of the second kind, which closes a box of the second kind. Next we look at a CCB of the first kind, which closes a box of the first kind opened in Stage 2 above.

1. This means 6-cyclic, and recalls that 6 = 2·3.

FIGURE 7-8.

$b = 1.78224$

The snap-back to a periodic SBR.



Stage 7: $b = 1.79032$

As in the preceding CCB of the second kind, this CCB of the first kind is characterized by the merging of critical points into a k-cycle, which was born repelling at the time of the opening of the box of the first kind. In this case, the affected cycle is the 3-cyclic attractor, β, born in Stage 2 above.

Figure 7-9 shows the 3-cyclic chaotic attractor, in the context of the graph of f^3, with the points α_3 and β_3 on the diagonal, at the moment of a CB of the first kind. Figure 7-10 is an enlargement of the region containing β_3 and α_3 in Figure 7-9. Here we can see that a critical point, c_5, has merged onto β_3, and is the terminus of a homoclinic orbit of critical points.

FIGURE 7-9.

$b = 1.79032$

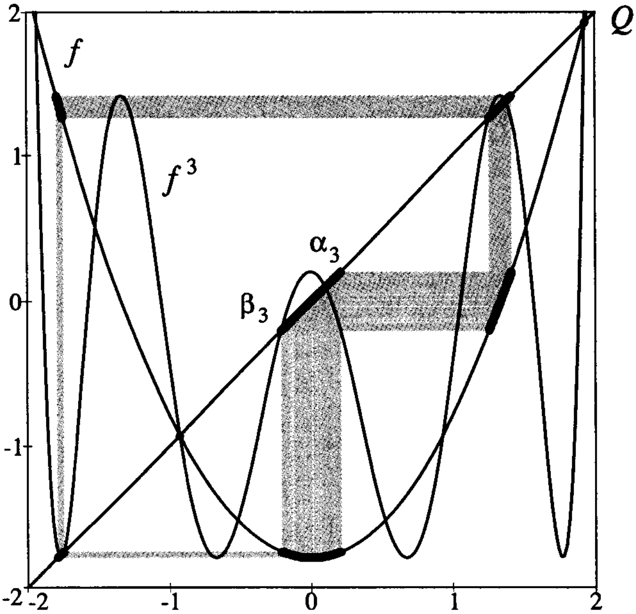
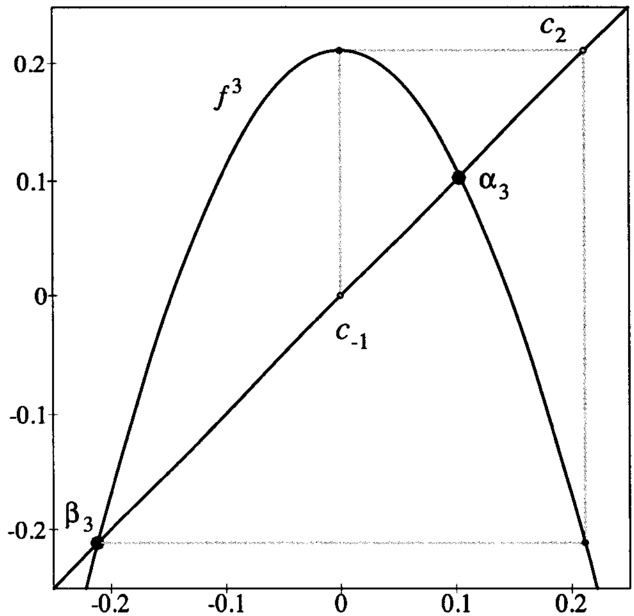


FIGURE 7-10.

$b = 1.79032$



Stage 8: *b* = 1.7905

Figure 7-11 shows that after a CCB of the first kind, the chaotic attractor has exploded into a single interval. Figure 7-12 is a view of the chaotic attractor alone, in the same frame. This is significantly different from the case of the CCB of the second kind described above. To explain this difference, consider again the fold bifurcation of Stage 2, in which the α and β 3-cycles were created, and in which the box of the first kind opened. Looking back at Figure 7-2, we can see that, at the moment of fold bifurcation, the points of the newborn cycle are attracting from one side (slope steeper than 1) while repelling to the other (slope less steep than 1). (This may be verified by drawing some cobwebs.) Still at the moment of this bifurcation, the new cycle has homoclinic orbits on the repelling side.

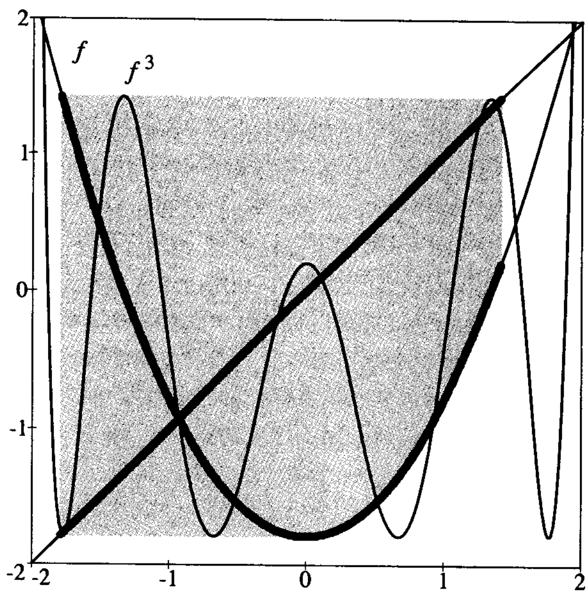
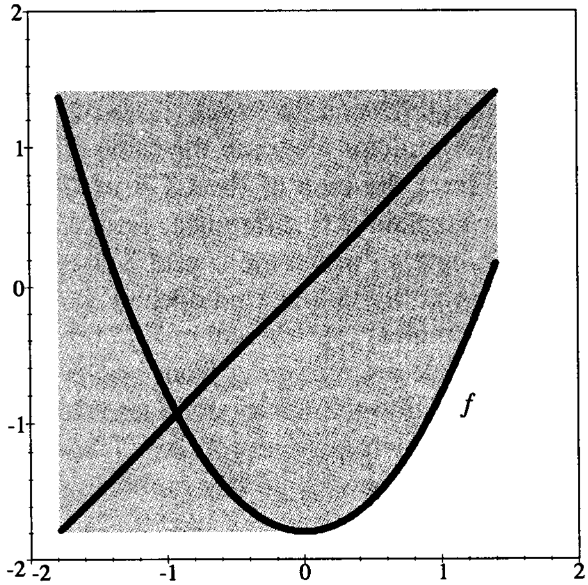


FIGURE 7-11.
$b = 1.7905$

FIGURE 7-12.

$b = 1.7905$

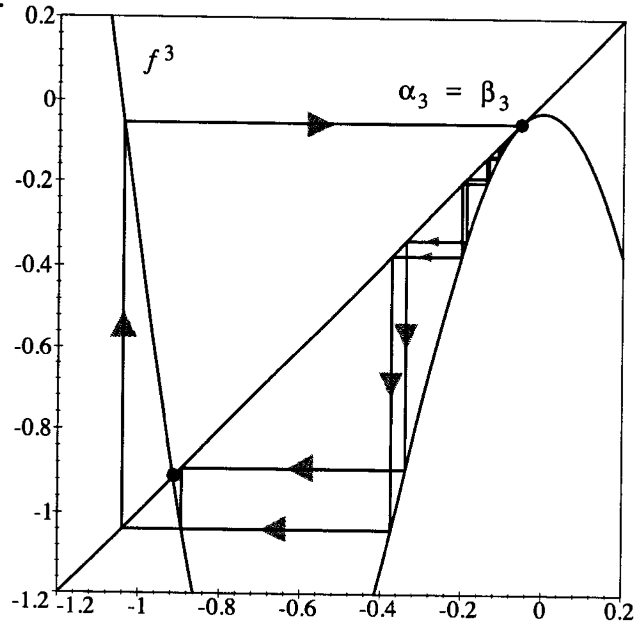


Stage 2: (again): $b = 1.75$

Figure 7-13 shows an enlarged view of the region near $\beta_3 = \alpha_3$ at the moment that point is created in the fold bifurcation. This point is repelling to the left, and two homoclinic orbits are shown. There are infinitely many such orbits. This point is homoclinic on one side only. Homoclinic orbits are also shown in Figure 7-14, *after* the fold bifurcation.

FIGURE 7-13.

$b = 1.75$



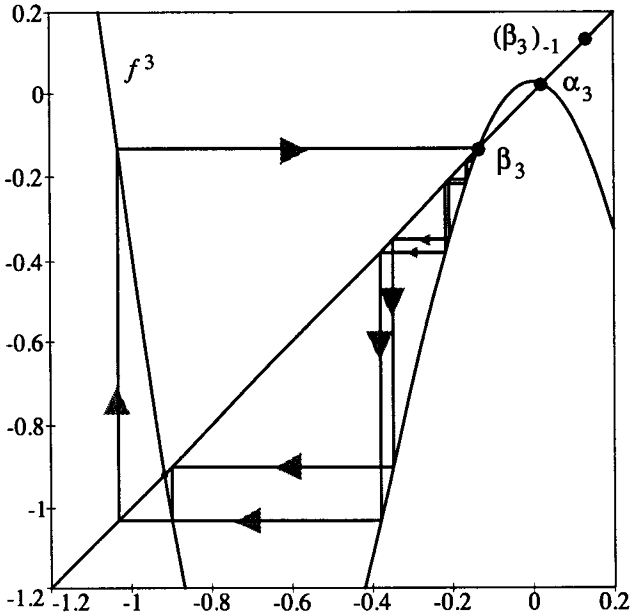
Stage 3: (again): b = 1.76

Looking back at Stage 3 in Figure 7-14, we see that β_3 still has infinitely many homoclinic orbits on the left, and none on the right. This means that the basin of attraction of the attractive 3-cycle, α, or of the cyclic absorbing intervals existing before the CCB of the first kind, is made up of the immediate basins together with all of their preimages, which have a chaotic, or fractal, structure. However, once a point of a trajectory enters the immediate basin bounded by β_3 and its preimage, $(\beta_3)_{-1}$, its images will enter the absorbing interval and never escape, as may be seen in Figure 7-8.

Summarizing the CCB of the first kind of Stage 6, the chaotic interval has a contact with β_3, and homoclinic points appear also on the other side of that point, as shown in Figure 7-10.

FIGURE 7-14.

$b = 1.76$

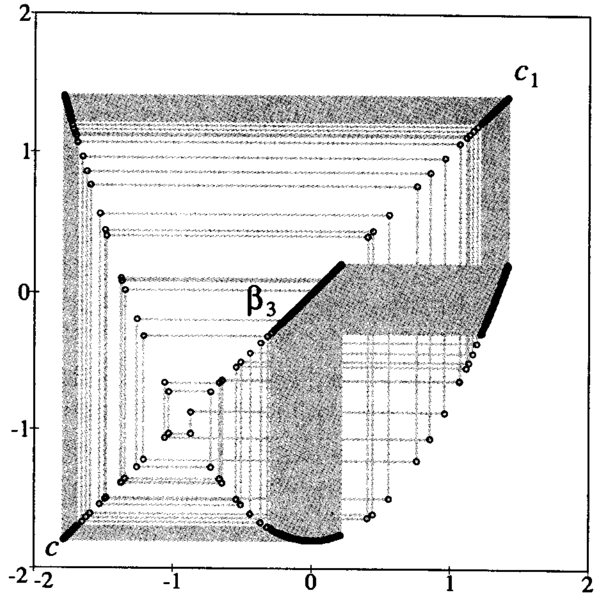


Stage 9: $b = 1.79035$

Figure 7-15 shows that after a CCB of the first kind (this does not occur in a CCB of the second kind) the generic trajectory of a point of the former immediate basin, $]\beta_3, (\beta_3)_{-1}[$, will escape from that open interval, covering the whole absorbing interval, $]c, c_1[$. In particular, soon after this CCB, the generic trajectory spends more time in the former chaotic intervals, but some rare moments outside them.

FIGURE 7-15.

$b = 1.79035$



7.2 BIFURCATIONS IN TWO DIMENSIONS

We now introduce chaotic contact bifurcations (CCBs), for maps of the plane. The introduction to CCBs in 1D of 7.1 will have provided a preparation for this material. The fine structure of CCBs may be easier to visualize in 2D than in 1D.

Context

Our context is a one-parameter family of maps of the plane into itself. Each map of the family is the generator of a discrete dynamical system based on the iterations of the map. As the parameter is changed, the dynamics (attractors, basins, and so on) of the map change. Special changes are known as bifurcations.

In this chapter we consider maps having a chaotic attractor. CCBs involve qualitative changes, or even destabilization, of the shape of a chaotic attractor. This has been a subject on the frontier of discrete dynamical systems theory since GM2 in 1978.

What is a CB?

A *contact bifurcation* involves a contact between the boundary of a chaotic attractor and the boundary of its basin of attraction. Both of the boundaries involved in the contact may be fractal. The underlying causes of fractal (rough) basin boundaries comprise a main theme of dynamical systems theory, and several mechanisms may be seen in the exemplary bifurcation sequences of this book.

In one of these exemplary sequences, we will explain, step-by-step, a bifurcation sequence taken from GARF. In this sequence, the CCBs correspond to homoclinic bifurcations of repelling cycles (nodes, foci, or saddles) of the map. In preparation for the detailed explanations of those examples, we will introduce here some of the basic concepts, and some of the technical jargon, of CCB theory.

Basic concepts and notations

This is an informal introduction; precise definitions may be found in Appendices 2 and 3. A subset of the plane is *invariant* under the map if the subset is mapped exactly onto itself. A *chaotic area* is an invariant subset (larger than a finite set) that exhibits *chaotic dynamics*, that is, a typical trajectory fills the area densely. A subset of the plane, A, is *attracting* if it has a neighborhood (an open set, U, containing A) every point of which tends asymptotically to A, or arrives there in a finite number of iterations. In this case, the basin or *basin of attraction* of A, denoted $D(A)$ or D, is the set of all points which eventually enter A; this basin may be found by taking the union of all preimages of the neighborhood U. The *immediate basin* of A, denoted $D_0(A)$ or D_0, is the largest connected part of D containing A. A *chaotic attractor* is a chaotic area which is attracting.

Consider now a map T with a chaotic attractor, d, its basin D, and the boundary of this basin, $F = bd(D)$, also called the *frontier* of d.

The attractor is called a *k-cyclic chaotic attractor*, where k is a positive integer, when d has k connected components which are permuted cyclically by the map T. Let T^k denote the map T applied k times in succession. Then each component of the attractor d of T is individually an attractor of T^k. In this case, the basin D is also k-cyclic. That is, it has k connected components, each being an image of the immediate basin, which are permuted cyclically by T. Each basin component, and each attractor component, are invariant under the map T^k. This applies to F as well.

What is a CCB?

A *chaotic contact bifurcation* occurs in this context when, as the parameter of the family is varied, the chaotic attractor d moves toward its basin boundary, the frontier F, and eventually makes contact when the attractor boundary, $bd(d)$, touches F. This frontier may be either a fractal, or smooth but the limit of an infinity of folded loops compressed by the action of the map. Also, the frontier contains repelling cycles, either saddles or repellors of nodal or focal type.

If d is k-cyclic, $k > 1$, we use T^k instead of T to visualize this event more easily. Let d_0 be one of the components of d. Then the attractor of T^k, d_0, will drift toward its basin boundary $F_0 = bd(d_0)$.

Classification of CCB types

A point P of F is an *isolated point* of F if it has a neighborhood, every point of which is not in F.

A *snap-back repellor* (SBR) is a repelling node or focus, P, which has a *homoclinic point*, that is, a point Q which has preimages approaching asymptotically to P, and also has an image which is P. That is, the homoclinic point Q comes from P in the infinite past, and arrives at P in the finite future. Hence, the orbit of Q is homoclinic, or "same-tending", in the sense of tending to the same point, P, in the past and in the future.

We classify CCBs as type I or type II. Those of type II are further divided into three kinds: first kind, second kind, and final.

In a *CCB of type I*, the contact of $bd(d)$ and F is first made at a fixed point, P, which is an isolated point of F until the moment of contact. Usually, this contact point becomes an SBR, in which case the CCB is called a *homoclinic bifurcation*. At the instant of a CCB of type I, the boundary touches an SBR, and the homoclinic orbits make their first appearance at P at the moment of contact. In this case, the point P is no longer within F at the moment of contact.

In a *CCB of type II*, the contact of $bd(d)$ and F occurs at points which were not isolated points of F before the moment of contact. These are also usually homoclinic bifurcations.

The three kinds of CCB of type II

A CCB of type II is *of the first kind* if it causes a sudden change in the shape of a chaotic attractor, such as a sudden change in size.

A CCB of type II is *of the second kind* if it causes a qualitative change in the shape of a chaotic attractor, such as the joining of a finite number of chaotic sets into a smaller number of chaotic sets, which continue to attract after the bifurcation.

A CCB of type II is a *final bifurcation* if it destroys the attractor, that is, it changes from a chaotic attractor to a chaotic repellor.

If F_0 is a limit set of components of basin boundaries of other bounded attractors (that is, consists of limit points of sequences of points belonging to basin boundaries) then a CCB of type II of the first kind may occur. If the contact points belong to the immediate basin of another bounded attractor, then a CCB of type II of the second kind may occur.

The simplest case in which a CCB of type II is a final bifurcation occurs when the contact points are limit points of the basin of infinity (the set $D(\infty)$ of points having unbounded trajectories). However, other similar interactions may give rise to this kind of bifurcation, in which $D(\infty)$ is not involved, but another basin, D', plays its role.

Next, we give examples, with abundant graphics, of CCBs of type II of the first and second kinds. Other examples may be found in G1, GARF, and FMG. Several examples of final bifurcations have already been described – see the bifurcation at the last stage described in each of the preceding three chapters.

7.3 EXEMPLARY BIFURCATION SEQUENCE

We have introduced the basic concepts and notations of CCBs in 1D in Section 7.1, and in 2D in Section 7.2. Here, we continue the 2D discussion with an exemplary bifurcation sequence, showing CCBs of type II of the first and second kinds.

We use a quadratic family of maps of the unit square, [0, 1] x [0, 1], called the *double logistic family*; this is EQ2 in 1.5, for more details, see GARF and FMG. As the bifurcation parameter increases from 0.6 to 1.0, this family exhibits features of a family of one-dimensional quadratic maps — the logistic family, equivalent to the Myrberg family of Section 7.1 — on the *diagonal*, the set Δ of points of the form (x, y) with $x = y$. This set is mapped into itself by every map of the double logistic family. Another special feature of these maps is a symmetry with respect to reflection through the diagonal. Also, each map has four fixed points: the points (0, 0) and (0.75, 0.75) on the diagonal, and two other points, P_1^* and P_2^*, which are mirror reflections through the diagonal.

Skipping over some simple bifurcations for low values of the bifurcation parameter b, we come to a type II CCB of the second kind between 0.64218 and 0.64219.

Stage 1: $b = 0.641$

Just before the bifurcation, the double logistic map has a 14-cyclic chaotic attractor (the dark curves in Figure 7-16), which belongs to an annular absorbing area (shown in white in Figure 7-16).

In the enlargement of Figure 7-17, we see a 7-periodic point of saddle type, V, between two nearby chaotic areas, shown with its local stable curve or inset, W^s, and its unstable curve or outset, W^u. To fix these features, we should think in terms of the map T^{14}. The inset of V is the frontier between the basins of the two chaotic attractors. Note that the two rays of the outset of V are attracted to the two chaotic attractors, and approximate their shapes.

FIGURE 7-16.

The basin of infinity in gray; the attractor is black.

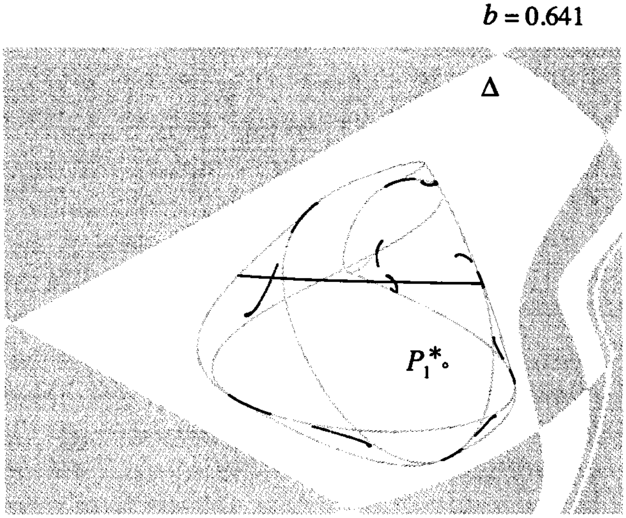
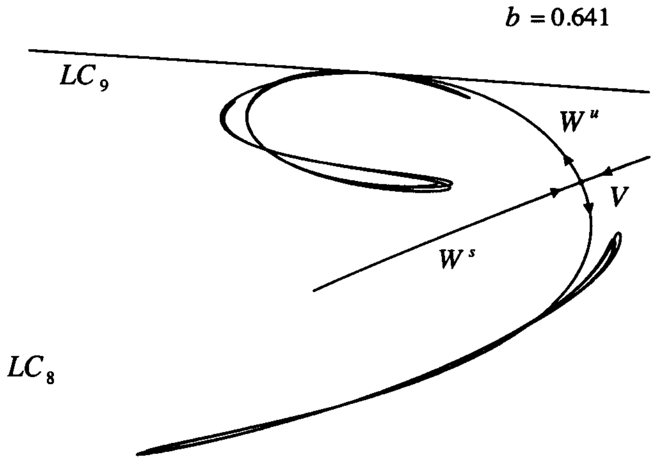


FIGURE 7-17.



Stage 2: *b* = 0.64218

Just an instant before the bifurcation, there is almost contact between the two chaotic areas and the frontier, the inset of *V*, as shown in Figure 7-18. Figure 7-19 is an enlargement of the small square in Figure 7-18. It shows how closely the chaotic attractors have approached to the frontier.

FIGURE 7-18.

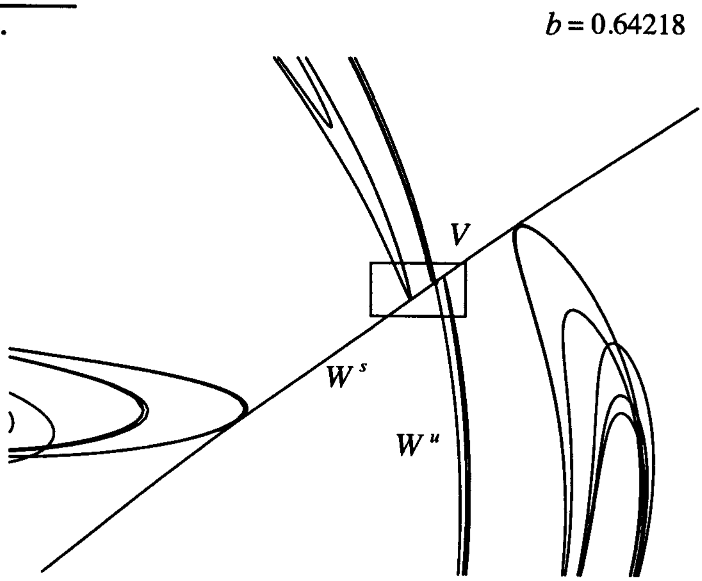
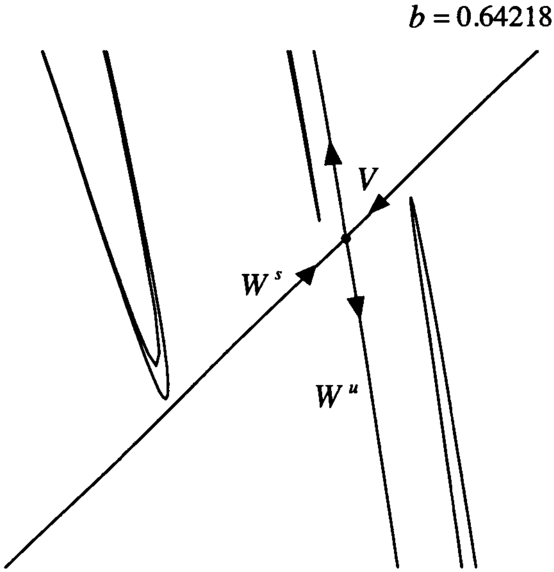


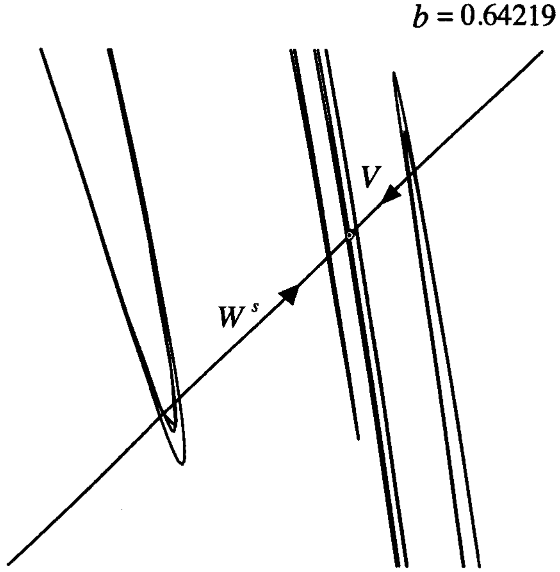
FIGURE 7-19.



Stage 3: $b = 0.64219$

Just an instant after the bifurcation, the chaotic attractors have passed through the inset of *V*, as shown in Figure 7-20. Because the outsets of *V* approximate these two attractors, which cross transversally through the inset of *V*, we may conclude that *V* is a transversally homoclinic saddle cycle: its outset has infinitely many transversal intersections with its inset. In fact, *V* has experienced a transition from nonhomoclinic to homoclinic state, exactly at our contact bifurcation.

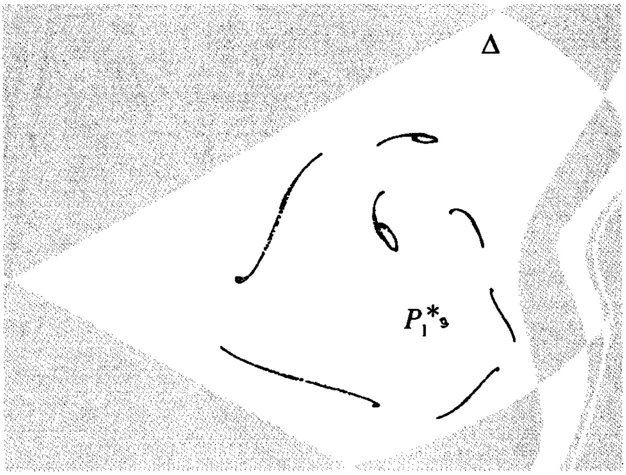
FIGURE 7-20.



Stage 4: *b* = 0.643

Later on, as shown in Figure 7-21, we see that the 14-cyclic chaotic attractor has become a 7-cyclic chaotic attractor, through the merging of pieces in pairs. This is characteristic of the CCB of type II of the second kind.

FIGURE 7-21.



Stage 5: *b* = 0.6439200

This stage represents the situation just an instant before a type II CCB of the first kind. In this event, the 7-cyclic chaotic attractor will explode into a larger, annular, connected (that is, 1-cyclic) chaotic attractor, which will be seen clearly later at stage 7. Such an explosion is characteristic of a type II CCB of the first kind.

In Figure 7-22, we see the complex dynamics just before this explosion. The attractor is shown in black, and the seven components of the basin of the 7-cyclic chaotic attractor are shown in 7 shades of grey, in Figure 7-22. (Please note: The seven components of one basin of the map T are seven distinct basins of seven distinct attractors of the map T^7.)

Figure 7-23 is an enlargement of the rectangle indicated in Figure 7-22, showing the close approach of a chaotic attractor of T^7 to its basin boundary. There is a 21-cycle of T of saddle type, that is, a 3-cycle saddle of T^7, belonging to the boundary of the immediate basin. This basin is shown in the lightest shade of grey.

FIGURE 7-22.

To each part of the attractor (black) is associated a corresponding piece of the basin (shades of gray.) One piece of the fundamental critical curve and two pieces of the basic critical curve are shown.

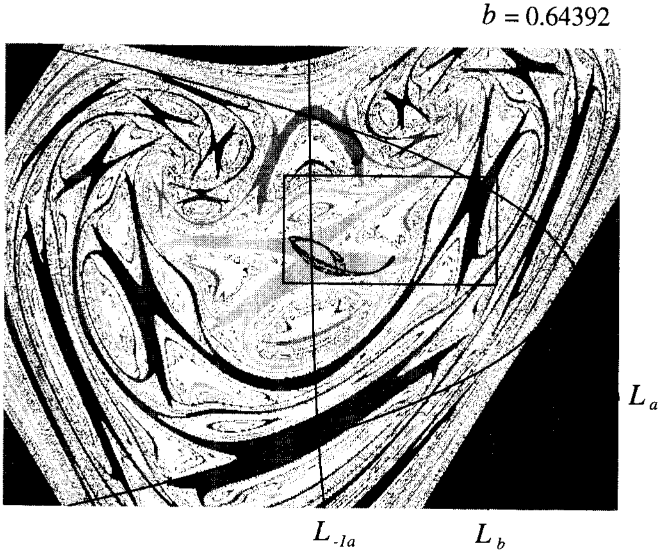
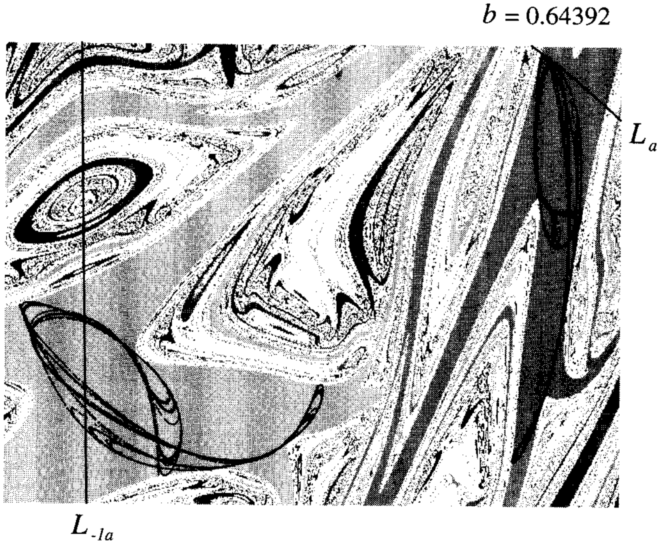


FIGURE 7-23.

Enlargement of the small rectangle above.



Stage 6: *b* = 0.6439248

Just after this contact bifurcation, the attractor has firmly pierced its former frontier. Figure 7-24 is an enlargement near one of the saddle points of the 21-cycle. This saddle lies on the former basin boundary, which is the inset of the saddle, and the attractor passes through this former frontier. Figure 7-25 is the same view, without the basins. Here we clearly see that the saddle has become homoclinic, as the unstable set crosses the inset of the saddle, and the chaotic attractor crosses the former frontier.

FIGURE 7-24.

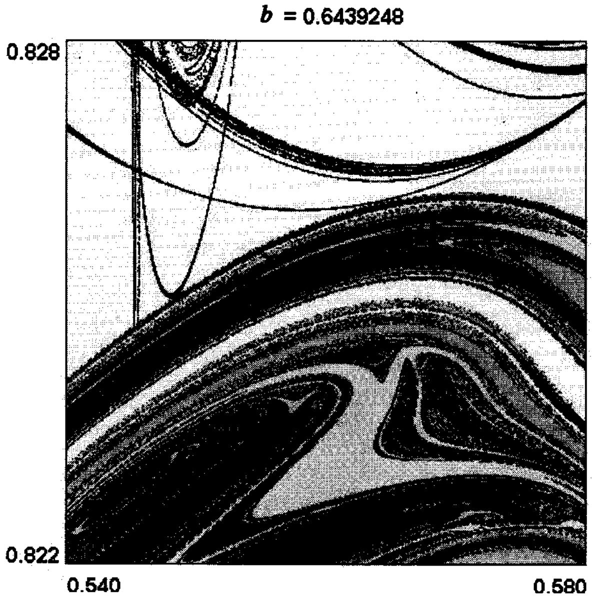
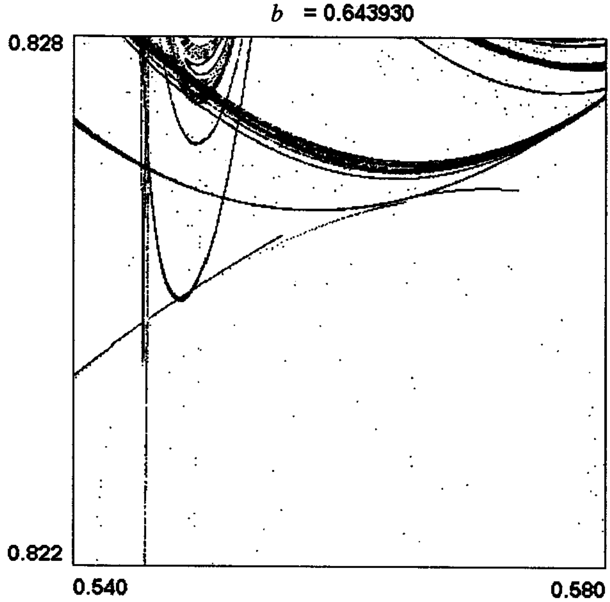


FIGURE 7-25.

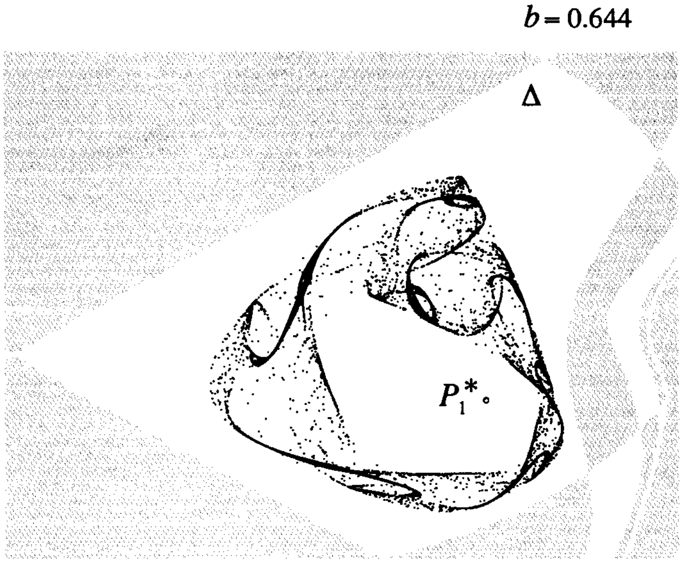


Stage 7: *b* = 0.644

Still later, we see the outcome of this CCB of type II of the first kind. Figure 7-26 shows the large, connected, annular chaotic attractor. It overlays the area formerly occupied by the seven pieces of the 7-cyclic chaotic attractor.

This large attractor persists until b reaches 0.702. Of course, it has a mirror image on the other side of the diagonal, due to symmetry, so there are actually two large chaotic attractors. The two corresponding immediate basins are separated by a segment of the diagonal, which segment consists of the inset of a 2-periodic cycle, $\{Q_1, Q_2\}$, which exists on the diagonal along with the fixed points of the map.

FIGURE 7-26.



Stage 8: $b = 0.7020$

A change suddenly occurs at this stage. The large annular chaotic attractors have expanded, and now make contact simultaneously with the diagonal. Figure 7-27 shows the points Q_1, Q_2, on the diagonal, Δ, and the contact just established with one of the large attractors. This is a CCB of type II of the second kind. The two attractors are becoming one, which will be symmetric, of course. At the bifurcation, the periodic points, Q_1, Q_2, are critical; that is, they belong to critical curves.

Stage 9: $b = 0.7025$

Just after the CCB, the outset of the 2-cycle, $\{Q_1, Q_2\}$ crosses the inset of the 2-cycle, which is in Δ, as shown in Figure 7-28. Thus, this 2-periodic saddle has become homoclinic during the CCB.

FIGURE 7-27.

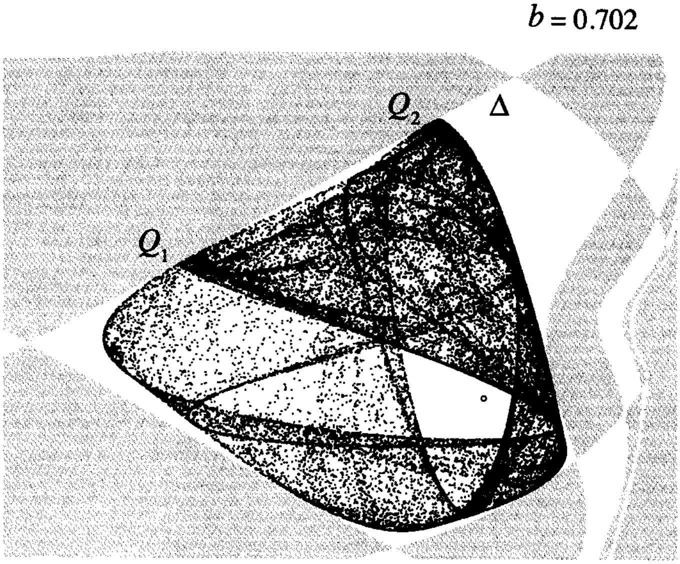
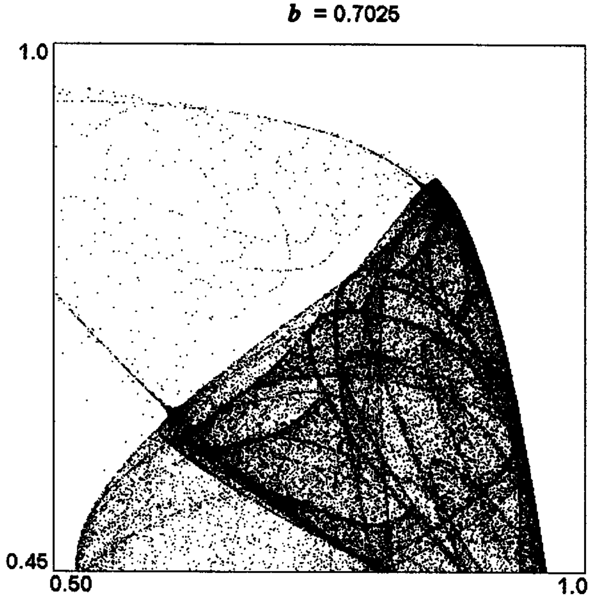


FIGURE 7-28.



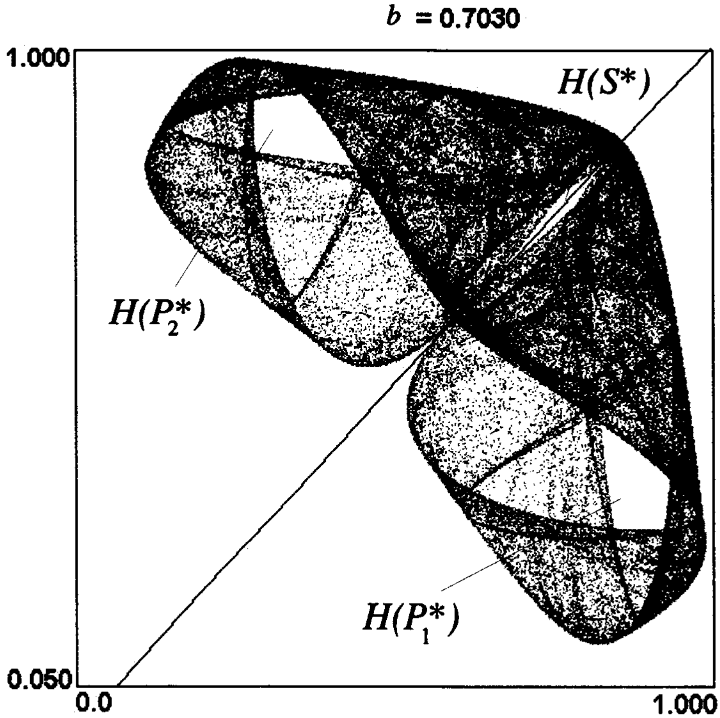
Stage 10: $b = 0.7030$

Figure 7-29 gives a global view of the new one-piece chaotic attractor. Note the three holes, labelled *H(S*)*, H(P_1*), and H(P_2*), which are bounded by critical curves. These surround the fixed repelling node on the diagonal, S^*, and the symmetric repelling foci, $P_1{}^*$ and $P_2{}^*$. These three holes will disappear at CCBs of type I, which are the first homoclinic bifurcations.

Stage 11: $b = 0.714$

At this CCB, the hole *H(S*)* disappears. Figure 7-30 shows the critical curves at this CCB. Figure 7-31 shows the attractor at the same moment.

FIGURE 7-29.



Stage 12: $b = 0.7375$

At this CCB, the holes $H(P_1{}^*)$ and $H(P_2{}^*)$ disappear. Figure 7-32 shows the critical curves at this CCB, Figure 7-33 shows the attractor, now simply connected, at the same moment.

FIGURE 7-30. $b = 0.714$

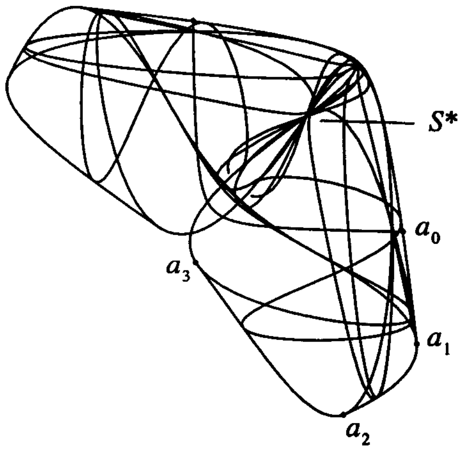
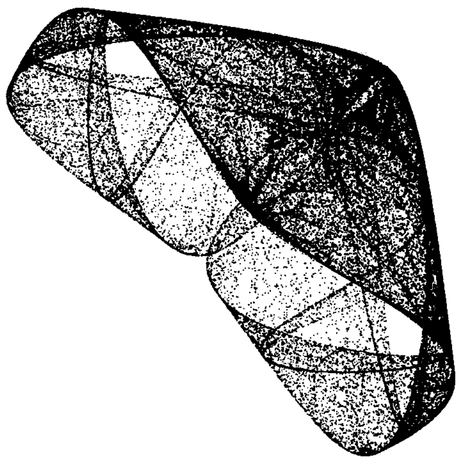


FIGURE 7-31. b = 0.714



As b continues to increase, several other CCBs of type I or type II may be observed, in which the sudden change of shape of a chaotic attractor is similar to the changes already seen in this sequence. Watch for them in the movie of the full bifurcation sequence provided on the CD-ROM which accompanies this book. We especially recommend $b = 0.88$, $b = 0.88498$, and $b = 0.88499$.

FIGURE 7-32.

$b = 0.7375$

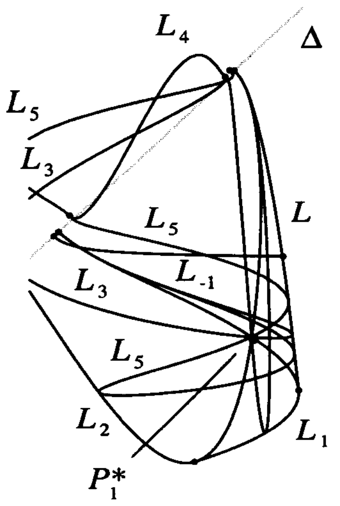


FIGURE 7-33.

b = 0.7375