Chaos in Discrete Dynamical Systems

A Visual Introduction in 2 Dimensions

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Companion CD-ROM Included,

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FOREWORD TO THE PROJECT

You are looking at the outcome of a three-year project, a unique experiment in electronic publishing. For lack of a better word, we call this a *package*. It has three intertwined components: a *book*, a *CD-ROM*, and a *website*. It is perhaps the first such multimedia package devoted to an advanced branch of mathematics.

The book is the primary component, and it is extensively illustrated with monochrome computer graphics. The CD-ROM is devoted mainly to 12 computer graphic animations in color, which animate and expand the graphics in the book. The user interface to the CD-ROM is made in the style, and with the technology, of the World Wide Web. Therefore, it integrates seamlessly with the website devoted to the book and CD-ROM, which is maintained at the Visual Math Institute. This website also connects outward with the resources of the World Wide Web.

The motivation for this unique package is the conviction that this style of electronic publication is the ideal medium for mathematical communication, and especially, for the branch of mathematics known as dynamical systems theory, including our subject: noninvertible discrete chaos theory in two dimensions. The essence of this communicative style is the *dynapic* technique, in which a drawing is developed stroke-by-stroke, along with a carefully coordinated spoken commentary. This is the traditional method used by most mathematicians, when speaking among themselves: *Visual Math*!

We will now introduce the three components separately.

PREFACE TO THE BOOK

This book is a visual introduction to chaos and bifurcations in noninvertible discrete dynamical systems in two dimensions, by the method of critical curves.

Historical Background

Dynamical systems theory is a classical branch of mathematics which began with Newton around 1665. It provides mathematical models for systems which evolve in time according to a rule, originally expressed in analytical form as a system of ordinary differential equations. These models are called *continuous dynamical systems*. They are also called *flows*, as the points of the system evolve by flowing along continuous curves.

In the 1880s, Poincaré studied continuous dynamical systems in connection with a prize competition on the stability of the solar system. He found it convenient to replace the continuous flow of time with a discrete analogue, in which time increases in regular, saltatory jumps. These systems are now called *discrete dynamical systems*. So, for over a century, dynamical systems have come in two flavors: continuous and discrete. Discrete dynamical systems are usually expressed as the iteration of a map (also called an endomorphism) of a space into itself. In these systems, points of the system jump along dotted lines with a regular rhythm.

In the context of a discrete dynamical system, in which a given map is iterated, that map might be *invertible* (because of being one-to-one and onto) or *noninvertible* (failing one or the other or both of these conditions). So, discrete dynamical systems come in two types, invertible and noninvertible. The invertible maps were introduced by Poincaré, and have been extensively studied ever since. The studies of noninvertible maps have been more sparse until recently, when they became one of the most active areas on the research frontier because of their extraordinary usefulness in applications.

Chaos theory is a popular pseudonym for dynamical systems theory. This new name became popular about 20 years ago, when its applicability to chaotic systems in nature became widely known through the advent of computer graphics. As there are two flavors of dynamical systems, continuous and discrete, there are also two chaos theories. The first to develop, in the work of Poincaré about a century ago, was the theory of chaotic behavior in continuous systems. He also studied chaotic behavior in discrete dynamical systems generated by an invertible map.

Discrete chaos theory for noninvertible maps began some years after Poincaré. Its development has been accelerated particularly since the computer revolution, and today it is a young and active field of study. The earliest development of the theory came in the context of *one-dimensional maps,* that is, the iteration of a real function of a single real variable. One of the key tools in the one-dimensional theory was the calculus of critical points, such as local maxima and minima. The *two-dimensional context* is the current research frontier, and, it is the subject of this book.

For two-dimensional noninvertible maps, the *critical curve* is a natural extension of the classical notion of critical point for one-dimensional noninvertible maps. The first introduction of the critical curve, as a mathematical tool for two-dimensional noninvertible maps, appeared in papers by Gumowski & Mira in the 1960s (see the bibliographies at the end of the book for references.)

The importance of our subject

Chaos theory generally is crucially important in all the sciences (physical, biological, and social) because of its unique capability for modeling those natural systems which behave chaotically. It is for this reason that there is a chaos revolution now ongoing in the sciences. For those systems which present continuous, evolving, data (such as the solar system) — continuous chaos theory provides models. And for those which present discrete data (such as economics) — discrete chaos theory provides models. One advantage of discrete dynamical models is the ease and speed of simulating the models with digital computers, as compared with continuous dynamical models. Discrete models are sometimes advantageous, even in the context of natural systems presenting continuous data.

Uniqueness of this publication

The book component of this book/CD-ROM/Website package is not a conventional text book, and yet its purpose is pedagogic. It intends to provide any interested person having a minimal background in mathematics, but with a basic understanding of the language of set-theory, to become an initiate in this new field. It is unique in providing both an elementary and a visual approach to the subject. While chaos theory is mathematically sophisticated, by focusing on examples and visual representations — there are about one hundred computer graphics in the book — and minimizing the symbols and jargon of formal mathematics — they are relegated to a set of appendices — the text provides the reader with an easy entry into this important and powerful theory. The primary focus of the package is the concept of *bifurcation* for a *chaotic attractor.* These are introduced in four exemplary bifurcation sequences, each defined by a family of very simple noninvertible maps of the plane into itself. Each family, the subject of an entire chapter in the book, exhibits many bifurcations.

And as dynamics involves motion, computer graphic animations provide a particularly appropriate medium for communicating dynamical concepts. The CD-ROM contains 12 animations which bring life to the basic ideas of the theory, literally animating the still images of the book. For each of the four map families there is one long, fast movie which is a fast forward through the entire chapter, as well as two “zooms” which expand a brief piece of the action into a slow motion movie. The movies can be understood only by reading along in the book while viewing the movie. The motion controls of the movie players (in both the Windows and the Macintosh environments) allow easy stop, play, fast-forward, reverse, and slow-motion, by dragging a slider. This makes the CD-ROM ideal for studying in conjunction with the book.

Intended audience

While many devotees of pure mathematics may enjoy this package for the novelty of its fresh ideas and the mathematical challenge of a new subject, with most of its main problems unsolved, the intended audience for this book is the large and heterogeneous group of science students and working scientists who must, due to the nature of their work, deal with the modeling and simulation of data from complex dynamical systems of nature which are intrinsically discrete. This means, for example, applied scientists, engineers, economists, ecologists, and students of these fields.

How to use the book

The book is divided in three parts, which are almost independent, and which can be utilized in parallel. The first part provides the simplest introduction to the basic concepts of discrete chaos theory, with many drawings and examples. The second part is a detailed analysis of computer experiments with four families of discrete chaotic systems, with emphasis on the method of critical curves, and the phenomena of bifurcation. The third part is a set of appendices which provide more official definitions for readers having a stronger background in abstract mathematics. Here, is also found extensive historical material by Professor Mira, some made available in English for the first time. It is proposed that the second part be regarded as a “guided tour” through a very difficult terrain, and each example studied repeatedly, with recourse as necessary (using the index) to the first and third parts, and to the CD-ROM.

ABOUT THE BOOK AUTHORS

Ralph Abraham is Professor of Mathematics at the University of California at Santa Cruz, founder of the graduate program there on computational dynamics, and is an author of

- *Foundations of Mechanics,*
- *Manifolds, Tensor Analysis, and Applications*, and
- *Dynamics, the Geometry of Behavior.*

Laura Gardini is Professor of Mathematics in the Universities of Urbino and Brescia in Italy, and is an author of

- *Chaotic Dynamics: Two-Dimensional Endomorphisms.*

Christian Mira is Professor of Control Engineering at the University of Paul-Sabbatier in France, is the founder of a laboratory of computational dynamics there, and is an author of

- *Dynamique chaotique: transformations ponctuelles, transition ordre-desordre,*
- *Recurrences and discrete dynamic systems,*
- *Chaotic Dynamics*, and
- *Chaotic Dynamics: Two-Dimensional Endomorphisms.*

As the creator of the method of critical curves, Christian Mira brings to this book long and extensive experience in the field. Laura Gardini extended the method of critical curves and applied it extensively, recently obtaining many new results. Ralph Abraham, known for his pioneering work — and his extensive book writing and illustrating — on continuous dynamical systems since 1960, met Mira and Gardini at a conference in June, 1991, and quickly became their co-author in this work.

PREFACE TO THE CD-ROM

The CD-ROM supplied in the back of the book is intended as an enhancement to the book. Its main function is to animate the graphics in chapters 4 through 7 with 12 movies. It also contains some useful software. This companion CD-ROM may be regarded as a "canned" piece of the World Wide Web. It has an index which may be accessed by any WWW browser, like Netscape Navigator, or Internet Explore. The CD also connects seamlessly with the Web, if your computer has Internet access.

The movies

The movies for chapters 4, 5, 6, and 7 are computer graphic animations, created by extensive computations with ENDO, an X-Windows software package for research on discrete dynamical systems in two dimensions created by Ronald Joe Record. These movies provide the best opportunity to understand the role of critical curves in the bifurcations presented in these chapters.

Each of the four chapters — 4, 5, 6, and 7 — present an exemplary bifurcation sequence. This means that we are given a one-parameter family of maps, and we carefully observe a chaotic attractor as the parameter is varied. Certain special events called bifurcations occur, perhaps very frequently, as the parameter is changed. In each of these chapters, we have singled out just a few of these special events for special attention, we call them "stages".

For example, in Chapter 4, there are 12 stages. In the book, monochrome computer graphics are included for each of these stages, along with extensive commentary which tries to explain the (very complicated) images.

In the movies, the stages are embedded in a very large number of in-between images, which are then flashed on the screen like a flip book. Thus, the still-frame black-and-white stage images of the book are embedded in an apparently continuous, uniform, sequence of color-coded images in the movies. The color code is a one-dimensional spectral scale from blue to red, and is shown at the right side of the screen in all of the movies. In the square frames of the movies, the color blue indicates a low relative density of trajectory points in a given small square of the plane, while red indicates a high density.

Additional CD-ROM content

Besides the twelve movies, each in two formats, the CD-ROM also contains additional material: MAPLE and ENDO.

The 96 computer graphics in chapters 4 through 7 of the book (with four exceptions) have been computed in the mathematical programming language MAPLE by Scott Hotton. For the 92 images that have been made by in this way, the complete programs (they are plain text files) may be read directly from the CD-ROM. Reading one of these files, with the help of a MAPLE programming manual if needed, answers all possible questions about the figures in the book: the size of the domain, the number of points, etc. In addition, the programs are very easily modified and run in the MAPLE environment, to do further research in chaos theory.

The ENDO program, written by Ron Record, was used by him to make all of the frames of the movies on the CD-ROM. It is an easy-to-use research environment which you might use to do frontier research in two-dimensional discrete chaos theory, if you have access to an X-Windows environment. We have included the complete program on the CD-ROM, in an archived and compressed UNIX file. Instructions for its installation are found in the file "index.html" on the CD-ROM.

Finally, the CD-ROM contains (in file "index.html") a few pointers to relevant websites, for those who have an Internet connection.

How to navigate the CD-ROM

There are two methods for accessing the CD-ROM.

Method #1. The first method, which we strongly recommend, makes use of a World Wide Web browser. The one we have used is Netscape Navigator, which is freely available on the Internet. All other browsers should work, but we have not tested them. In this method,

A. Insert the CD-ROM in the CD-ROM drive.

B. Start the browser.

C. Click the File item on the browser menu bar.

D. Choose the “Open File” option

E. Browse to the file “index.html” on the CD-ROM.

F. Open it.

Then all contents of the CD-ROM are displayed for your choice. This is particularly convenient for the MAPLE script files. Also, if you happen to be connected to the World Wide Web, you may click on some links to external servers.

Note: Clicking on a movie in the web browser results in a one-minute wait, while the movie file is copied from the CD-ROM to your hard disk. This is bad, because you have to wait. On the other hand it is good, because the movies play better from the hard disk, unless your equipment is in perfect running order. After the wait, you will see the first frame of the movie in the web browser window. You may then start and stop the movie by clicking anywhere in its frame.

Method #2. This is the fall-back method, and does not require any software other than the Windows FileManager, Macintosh Desktop, or UNIX shell.

A. View the contents of the CD-ROM.

B. Double click on the item of choice.

Because this CD-ROM is a hybrid CD, the file structure looks like Windows to Windows, looks like Macintosh to Macintosh, and looks like UNIX to UNIX.

Hardware and software requirements

The 12 movies are each provided in two formats on the CD-ROM: AVI and QuickTime. Both are 320x240x8 video with 22kHz by 16 bit sound. On a Macintosh you must use the QuickTime versions. Under Windows you would choose the AVI version, unless you have QuickTime for Windows on your system, in which case you have a choice. QuickTime for Windows is available from Apple over the Internet, and our CD-ROM has a link to Apple to help you obtain a copy. In any case, you may play the movies through the web browser, as described above in the preferred Method #1. On the other hand, with the fall-back Method #2, the QuickTime movies may be played with the Movie Player included in the Macintosh operating system, while the AVI movies may be played with the MediaPlayer which is part of the Windows operating system.

These movie players have a simple control panel with run and pause buttons. In addition, you may drag the slider to advance or reverse the movie at slower or faster than normal speeds. You may use either format on a UNIX platform, with appropriate software, such as the freeware “xanim” for X-Windows. On Windows or Macintosh machines, you may also use a World Wide Web Browser to play the movies, as we have explained above.

The movies assume that your computer is capable of playing QuickTime (MOV) or Video for Windows (AVI) movies at 2X speed, that is, at 300 KB per second. If the movies jerk or stick, that probably means that your computer needs a tune-up.

Bugs

Every hardware/software platform plays CD-ROMs differently, and we cannot anticipate all of the potential problems. We have tested our CD-ROM on several machines of each sort — Windows, Macintosh, and UNIX. All functions have been robust and correct except the movie service function.

On older versions of Windows and Macintosh operating systems, the movie players seem to stick inconsistently. As a work-around, try moving the slider back and forth to loosen things up. Some older systems display a warning message upon first inserting the CD-ROM in its drive, but <RETURN> seems to work.

Here are some tricks to improve Macintosh movie performance. *Virtual Memory*: Typically, this is set on, and to about 1MB more than the actual RAM. For example, with actual RAM 16 MB, set virtual RAM to 17 MB. *Cache Memory*: This may be reduced to improve movie playing. *MoviePlayer application memory*: Increase the amount of memory devoted to MoviePlayer if you know how.

ABOUT THE CD-ROM AUTHORS

Ralph H. Abraham created the computational dynamics program at the University of California at Santa Cruz.

Ronald Joe Record is a Ph.D. graduate of the computational dynamics program at the University of California at Santa Cruz, and now works as a software engineer in Santa Cruz.

PREFACE TO THE WEBSITE

All of the material currently available is found in the book, or on the CD-ROM. However, upon publication of this package, additional graphics, questions and answers, will be posted on the web site devoted to the project and administered by the Visual Math Institute. We will maintain a Chaos FAQ (Frequently Asked Questions) and bug reports on the site, and other features which may prove useful to the international chaos community.

The URL for the web site is: http://www.vismath.org/chaos/jpx

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A video by John Dorband of NASA Goddard Space Flight Center originally sparked our collaboration presented in Chapter 7 here, and we are grateful to him for sharing his work with us.

Figures 7-24, 7-25, 7-28, and 7-29, very difficult to compute, are the work of our colleague Danièle Fournier-Prunaret, and we are grateful to her for contributing them to this book. Her beautiful drawings have inspired us.

Finally, we are very grateful to Peter Broadwell for the loan of a Silicon Graphics Indigo computer, which ran continuously for several *months* cranking out the frames for the movies on our CD-ROM. And without our mathematical copy editor, Paul Green, this book would be a mine field for the novice reader. We are deeply in his debt.

Dedicated to

Igor Gumowski