CHAPTER 1

INTRODUCTION

1.1 BACKGROUND

Our goal is to present the fundamentals of two-dimensional (2D) iteration theory through examples, with extensive graphics (for which the 2D context is ideal) and few mathematical symbols.[1] We illustrate all the basic ideas with hand drawings and monochrome computer graphics in the book, and again with movies (full-motion video animations in color) on the companion CD-ROM.

We do not assume a knowledge of higher mathematics. But we do acknowledge that our subject is a branch of pure mathematics, and a deeper understanding requires some topology and geometry. A hint of this is presented in the appendices, where a more rigorous approach is introduced.

1.2 HISTORY

The study of chaos in 1D iterations is a classical subject, going back to Poincaré over a century ago, as described in detail in Appendix 5. The 2D case (two real variables or one complex variable) also goes back almost a century but the stream of literature to which this book belongs really begins with the computer revolution and the pioneers of scientific computation — Von Neumann, Ulam, and so on — in the 1950s. Our subject remains an experimental domain, and computer graphic experiments provide our main orien-

1. This approach was developed in (Abraham, 1992).

tation. Our fundamental tool for describing the behavior of 2D iterations, the *critical curve*, was introduced by Gumowski and Mira in 1965.[1]

1.3 PLAN OF THE BOOK

In Part 1, we introduce the basic concepts and vocabulary of iteration theory, first in 1D, then in 2D. We try to introduce only as much theory as is required to understand Part 2, on exemplary bifurcation sequences. In Part 2, we will use the vocabulary and ideas of Part 1 to explain step-by-step the events in the exemplary bifurcation sequences.

We use the critical curves to understand the structure of attractors, basins, basin boundaries, and their bifurcations. Then we increase the bifurcation parameter, and explain the changes in the configuration of attractors and basins due to bifurcations of various types, again using the critical curves.

These structures and changes are illustrated with still images created by our software for the iteration of a fixed endomorphism, based on the method of critical curves. These graphics are strung together as movies, which may be viewed from the accompanying CD-ROM, to give a more dynamic idea of the sequence of bifurcation events in each of the exemplary families.

1.4 CONTEXT

Dynamics is a vast subject, and our subject is a relatively new frontier within it. So, for those who already have an idea of the territory of dynamical systems, we would like now to locate our subject within this larger territory.

Dynamical systems theory has three flavors:

1. See (Gumowski and Mira, 1965) and (Mira, 1965) in the Bibliography.

- *flows* are continuous families of invertible maps generated by a system of autonomous first-order ordinary differential equations, and parameterized continuously by time, that is, by real numbers;
- *cascades* are discrete families of invertible maps generated by the iteration of a given invertible map, and parameterized discretely by the integers (zero, positive, and negative);
- *semi-cascades* are discrete families of maps generated by iteration of a given map, generally noninvertible, and parameterized discretely by the natural numbers (zero and the positive integers).

Both cascades and semi-cascades are also known as *discrete dynamical systems*, or *iterations*. In this book we are primarily interested in semi-cascades generated by a noninvertible map, (NIM). For simplicity, we will simply call these iterations in future; but keep in mind that all of this book belongs to the NIM flavor.

In general, the *state space*, the space in which a dynamical systems is defined, may be an arbitrary space of any dimension: 1, 2, 3, and so on. This suggests a tableau of types of dynamical systems, as shown in Fig. 1-1. In this tableau, there is a relationship between cells on the same diagonal (marked with an A): In each row, the marked cell is the cell of lowest dimension in which *chaos* occurs. Hence, the tableau is called the *stairway to chaos*. Here chaos means any dynamic behavior more complicated than periodic behavior.

In this book, we discuss only the iteration of noninvertible maps, and the only state spaces we consider are the one-dimensional Euclidean line and the two-dimensional Euclidean plane. In fact, the latter is our primary subject. The 1D case has been extensively treated in recent literature (see M1) and shares the stairway to chaos with 2D cascades and 3D flows, the contexts for the early history of chaos theory. (See Appendix 5.) The second diagonal, marked with B here, may be regarded as the current frontier of chaos theory.

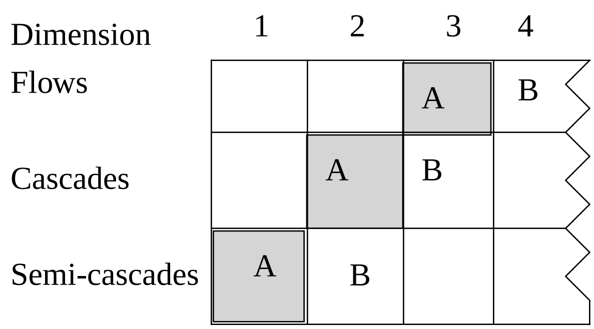

FIGURE 1-1.

The stairway to chaos.

1.5 BASIC CONCEPTS OF ITERATION THEORY

This section introduces the basic terminology. These concepts will be explained in detail in 1D and 2D in the next two chapters.

Iterated map: An iterated map is the generator of a discrete dynamical system; generally a noninvertible, continuous map.

Multiplicity: Our maps are usually *noninvertible*, that is, many-to-one, so a given point may have several preimages. The range set may be decomposed into with zones of constant multiplicity (bounded by *critical points* or *critical curves*) in which all points have the same number (called the *multiplicity*) of preimages. These multiple preimages determine a tree of *partial inverses* for the map. Multiplicities (explained further in the next chapter) play a fundamental role in our theory, analogous to the degree of a polynomial function.

Critical sets: These sets are boundaries of zones of constant multiplicity; thus, they separate zones of different multiplicity. They consist of points with *coincident inverses*.

Zones: The zones of constant multiplicity play a very fundamental role in our view of NIM theory, analogous to the role of degree of a polynomial in algebra.

Partial inverses: By restricting our attention to a zone of constant multiplicity in the range of a map, say multiplicity k, we may define k inverses to the map. These partial inverses play the role, in the NIM context, of the inverse of an invertible map.

Trajectory: A trajectory embodies the basic data of a dynamical system. It consists of the list of locations of the images of a particular point, called the *initial point*, under the iterations of the map generating the dynamical system. It is an ordered sequence (as opposed to a set) of points.

Attractors, basins, boundaries: These are the chief characteristic features of an iteration, from the qualitative point of view. Attractors are limit sets of trajectories of initial points filling an open set, which is the basin of the attractor. The boundaries of these basins are particularly important in applications of the theory.

Portrait: The state space of a dynamical system may be decomposed into a set of open sets (basins), in each of which is a single attractor. The boundaries of these basins are particularly important in applications of dynamical systems theory.

Bifurcations: These are fundamental changes in the qualitative behavior of a dynamical system, occurring as a control parameter is varied. At certain critical values of the parameter, the qualitative behavior of the trajectories of the system suddenly changes in a significant way. These sudden changes, called bifurcations, usually occur in sequences, called *bifurcation sequences*.

1.6 THE FAMILIES OF MAPS

The first family of maps we use to illustrate the basic ideas of discrete dynamics is from a paper of Kawakami and Kobayashi, studied also in a paper of Mira and coworkers:

$$u = ax + y$$

$$v = b + x^2 \qquad \text{EQ 1}$$

Usually, we fix the value of *a*, and vary *b* to exhibit a *bifurcation sequence*. We make use of three special cases in Part 2:

- Case 1: a = 0.7 (Chapter 4, Absorbing Areas)
- Case 2: a = 1.0 (Chapter 5, Holes)
- Case 3: a = -1.5 (Chapter 6, Fractal Boundaries).

These parameter ranges are shown in Fig. 1-2.

In Chapter 7, Chaotic Contact Bifurcations, we use the double logistic map studied by Gardini and coworkers,

$$u = (1-c)x + 4cy(1-y)$$

$$v = (1-c)y + 4cx(1-x) \qquad \text{EQ 2}$$

with *c* in [0, 1].

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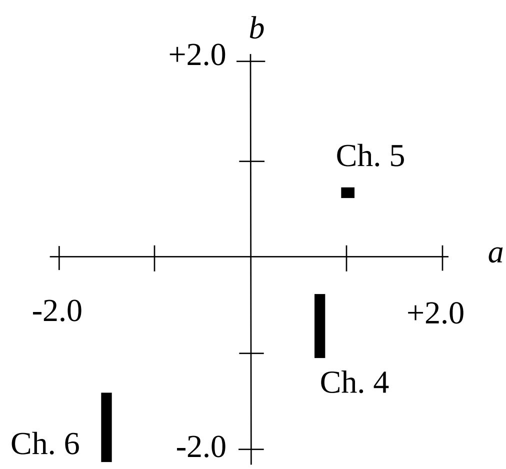
FIGURE 1-2.

Parameter space of the first family.



1.7 CRITICAL POINTS AND CURVES

The theory of critical curves for maps of the plane provides powerful tools for locating the chief characteristic features of a discrete dynamical system in two dimensions: the location of its chaotic attractors, its basin boundaries, and the mechanisms of its bifurcations. We next introduce the basic concepts of this theory first in 1D, then in 2D.