CHAPTER 2

BASIC CONCEPTS IN 1D

In the preceding chapter we introduced a brief list of basic concepts of discrete dynamics. Here, we expand on these concepts in the one-dimensional context, in which, uniquely, we have the advantage of a simple graphical representation. The official, abstract definitions of all these concepts may be found in the Appendices.

2.1 MAPS

By a *map* we mean a continuous function from a space, called the *domain*, to itself.[1] In the one-dimensional context, the domain might be an interval (with or without endpoints) of the real number line, or even the entire line.

If f is a map on a real interval, I, we indicate this in symbols $f : I \to I$. For example, if the map is defined by the rule $f(x) = x^2$, and I is the closed interval [-2, 2], we may visualize the map graphically, as shown in Fig. 2-1. The action of the map is to move points from the horizontal axis to the vertical axis in two strokes:

- vertically from the horizontal axis to the graph,
- horizontally from the graph to the vertical axis,

as shown in Fig. 2-2.

The *image* (or *range*) of the map is the set of all points obtained as $f(x)$ while x takes on all values in the domain, I, and is written in

1. The word *continuous* belongs to the branch of math known as *point-set topology*. This wonderful subject is not known as well as it ought to be, but nevertheless, we must use it constantly.

symbols as $f[I]$. For a point y in I, a *preimage of rank 1* is a point x in I that is mapped to y; that is, x is a preimage of rank 1 of y if $y = f(x)$. A *preimage of rank 2* of y is a preimage of rank 1 of a preimage of rank 1, and so on. Every point y in I has a set of preimages of every rank, which may be empty. Determining all preimages of a point creates a genealogical tree, called the *arborescent sequence of preimages*.

The map f is *one-to-one*, if, for every point y in I, the set of preimages of y has either no points or just one point. For example, the map f defined by the same rule, $f(x) = x^2$, but with the smaller domain [0, 1], is one-to-one. A one-to-one map has a unique inverse map, $f^{-1}: f[I] \to I$ which undoes what f does. This can be visualized on the graph of f as a motion in two strokes:

- horizontally from the vertical axis to the graph of f,
- vertically from the graph to the horizontal axis,

as shown in Fig. 2-3.

Note that if we try to invert the map of Fig. 2-2 by this two-stroke process, we discover all of the preimages of a given point y in I (represented as the vertical axis) in one step. This is shown in Fig. 2-4, where we find two preimages.

In this text we will be concerned exclusively with the lowest step of the staircase to chaos, the 1D case. We will be interested especially with maps which are not one-to-one. These are called *many-to-one*, or *noninvertible*, maps. For such maps, points generally have more than one preimage of rank 1, and the number of preimages of a given rank determines a zone of multiplicity, discussed below.

2.2 MULTIPLICITIES

Given any map of an interval I, we may choose a point y in I on the vertical axis, locate all preimages by the graphical method, and count them up. Thus, we may decompose the vertical axis into sets

FIGURE 2-1.

Graph of *f* on $I = [-2, 2]$.

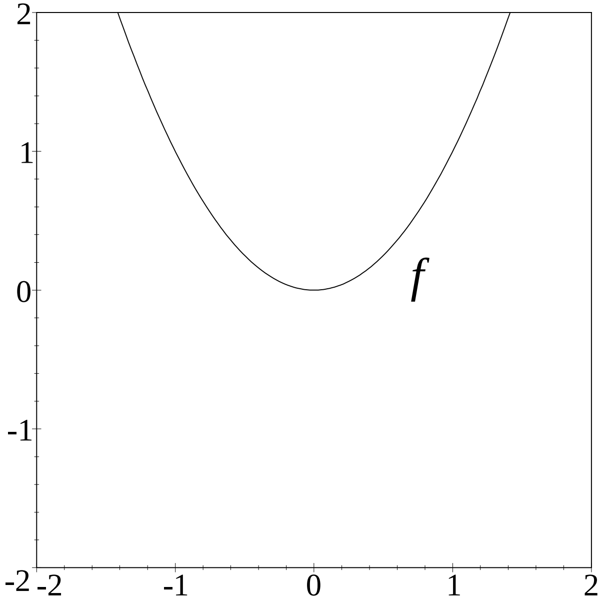
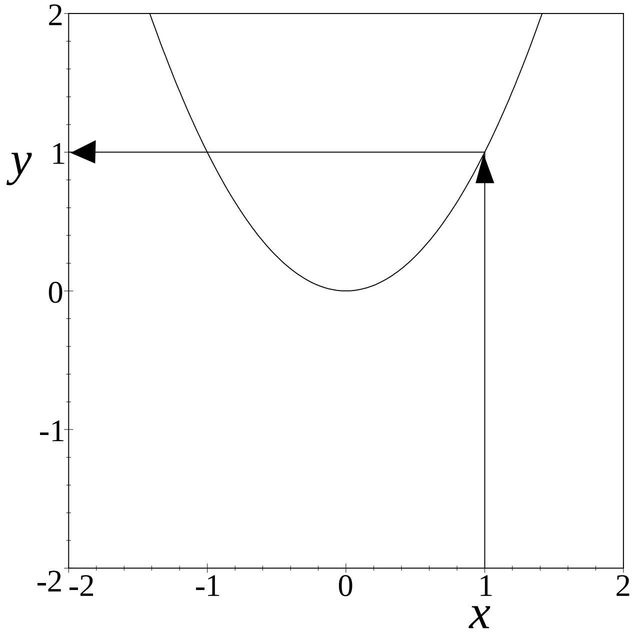


FIGURE 2-2.

Two strokes from a point x to its image y.



of points all sharing the same number of preimages. We denote these zones by:

- Z_0 (all points having no preimages)
- Z_1 (all points having exactly one preimage)
- Z_2 (all points having two distinct preimages)

and so on.

These are called the *layer sets* of the map; those sets that are nonempty and open (that is, have no boundary points) are called *multiplicity zones*. The zones Z_0 and Z_2 are shown in Fig. 2-5. They exhaust the whole interval $I = [\_2, 2]$ except for the point 0, which separates the two zones. We also say this map is *of type* $Z_0 - Z_2$, meaning there are two zones, one of multiplicity zero, the other of multiplicity two. The suffix indicates the multiplicity. The interval Z_2, may be considered a folded image, that is, two halves of the domain I are folded onto this image. For each half of the domain, our map does have an inverse. These are called *partial inverses*.

The point 0 is the only point of Z_1 in this example; that is, it has multiplicity one (its unique preimage is 0). It is called a *critical point* because it lies on the boundary of two zones. In fact, we can describe this map as a *nonlinear folding*. That is, the map folds the horizontal axis at the critical point, then stretches them in a nonlinear fashion onto the range interval. This is why the critical point is sometimes called a *fold point*.

Generally, we will be interested in relatively simple maps, such as polynomials, in which only finite multiplicities, with generic (that is, typical) fold points, are encountered. We call these *finitely folded maps*. For example, a typical cubic map has multiplicities 1 and 3, and we say it is of type $Z_1 - Z_3 - Z_1$. These basic concepts of iteration theory should be approached through a careful study of simple (for example, polynomial) examples.

FIGURE 2-3.

Two strokes from a point y to its preimage of rank 1.

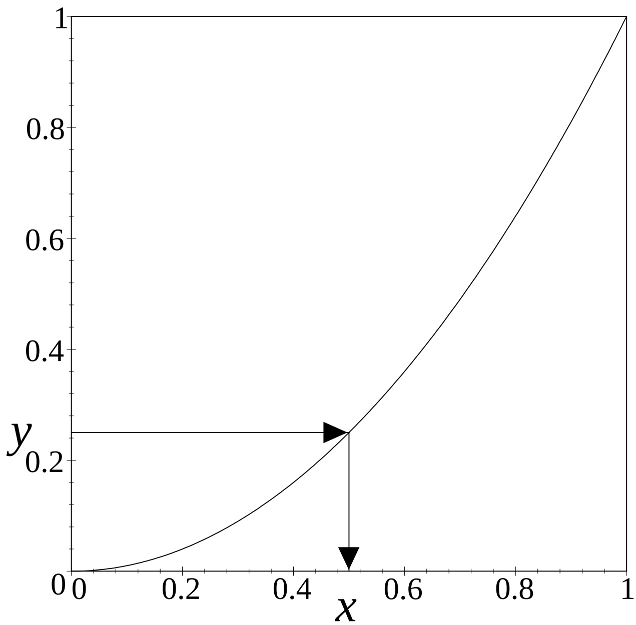
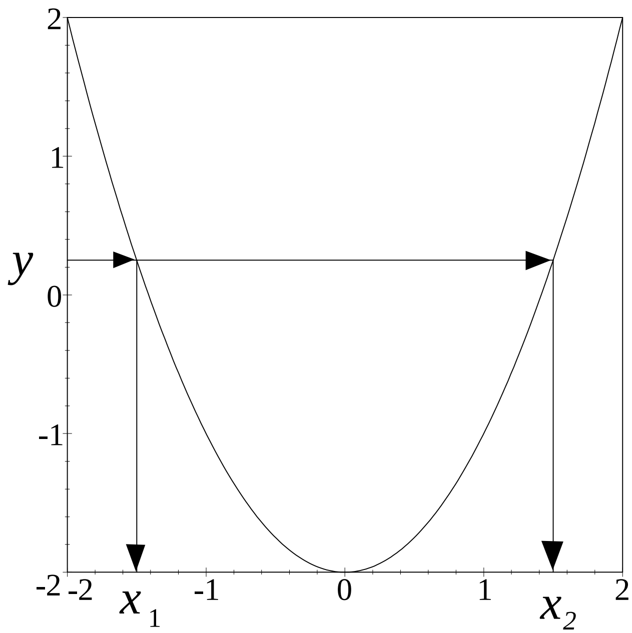


FIGURE 2-4.

Finding all preimages of rank 1 of a point y.



2.3 TRAJECTORIES AND ORBITS

A map generates a discrete dynamical system by iteration. That is, the map is applied again and again, and points move along a dotted path called a *trajectory*. For example, choosing an *initial point* x_0, let x_1 denote its image under the map *f*, likewise x_2 the image of x_1, and so on. The infinite sequence $(x_0, x_1, x_2,...)$ is the trajectory of x_0. This sequence may jump around a finite set of points. The minimum set of points which holds a trajectory its called its *orbit*. When finite, an orbit is called *cyclic*, or *periodic*, and the number of its points is its *order*. A *fixed point*, defined by $f(x) = x$, is a special kind of periodic point, the orbit of which is a single point. It has order 1. If an orbit contains only two points, it is called a *2-cycle*, and so on.

We now describe a graphical method for plotting trajectories, called the *Koenigs-Lemeray method*. Note that in our graphs, both axes represent the same set, since the domain and range of our map consist of the same interval.

Given x_0, envisioned on the horizontal axis, we may find x_1 on the vertical axis by the two-stroke method described in 2.1. Next, we must repeat this process, starting from the point x_1 on the horizontal axis. Our immediate problem, then, is to transfer the distance x_1 from the vertical axis to the corresponding distance on the horizontal axis.

One way to carry out this transfer is shown in Fig. 2-6. Here we use a compass to measure the vertical distance, x_1, and rotate it to the horizontal distance, x_1. Another method is to use a protractor to construct a line descending at slope -1, or 45 degrees, as shown in Fig. 2-7.

Yet another method — and this is the one we prefer — is shown in Fig. 2-8. We draw a line from the lower left corner, ascending at slope 1. This line is called the *diagonal* (in symbols, Δ). Now, using only a square, we draw a horizontal line from x_1 on the vertical axis until it meets Δ, then draw a vertical line until it meets the horizon-

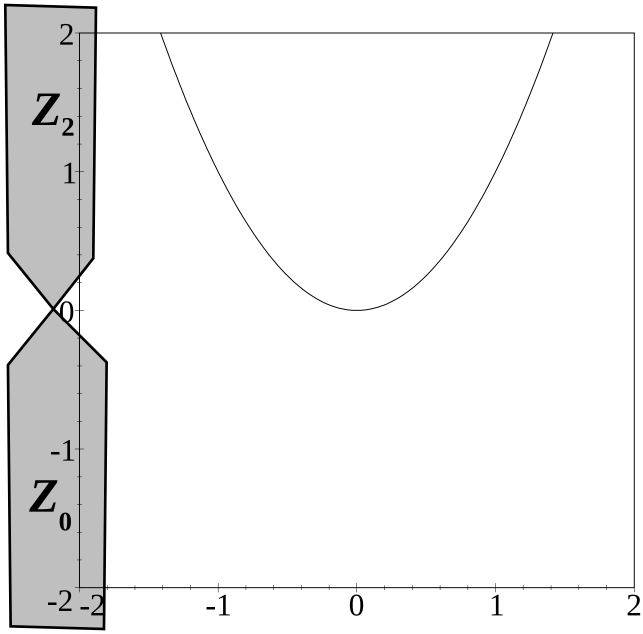


FIGURE 2-5.

The multiplicity zones for a quadratic function on [-2, 2].

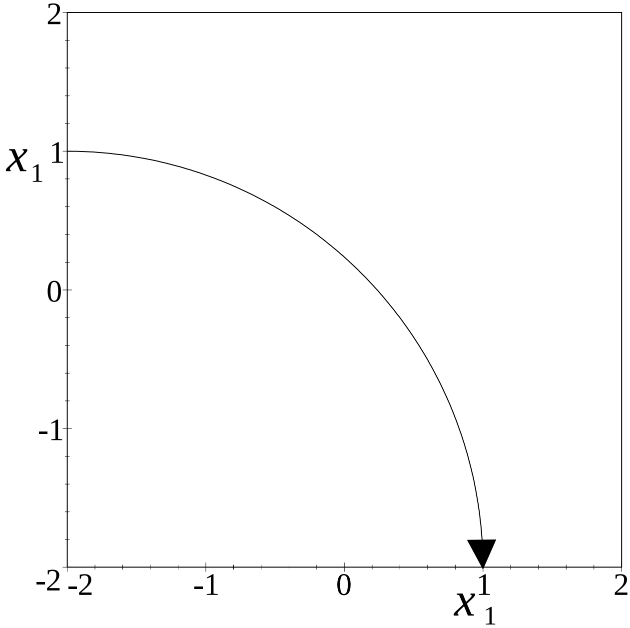


FIGURE 2-6.

The compass construction.

FIGURE 2-7.

The descending line method.

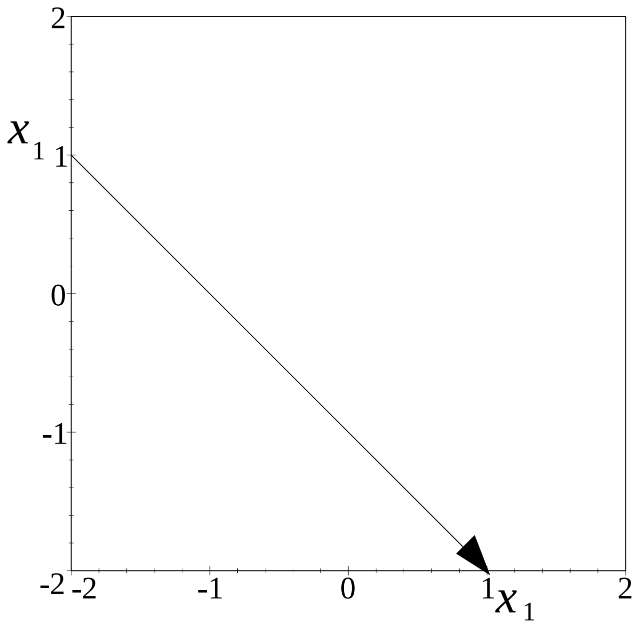
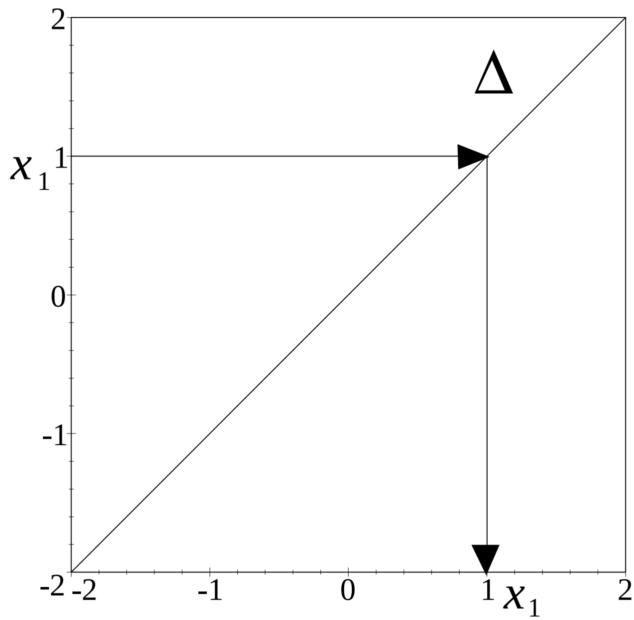


FIGURE 2-8.

The square two-stroke method.



tal axis. This determines the horizontal distance, x_1, as shown in Fig. 2-8.

The entire construction from horizontal x_0 to vertical x_1 to horizontal x_1 may now be summarized as follows:

- vertical from horizontal axis to graph,
- horizontal from graph to vertical axis,
- horizontal from vertical axis to diagonal,
- vertical from diagonal to horizontal axis.

This construction may be abbreviated somewhat since the third stroke retraces (undoes) part of the second, as shown in the four strokes of Fig. 2-9. The abbreviated construction (Fig. 2-10) is:

- vertical from horizontal axis to graph,
- horizontal from graph to diagonal,
- vertical from diagonal to horizontal axis.

This is the three-stroke graphical method for plotting one step of a trajectory within the horizontal axis. When we proceed to plot the point x_2 by this method, however, we find a further opportunity for abbreviation. The last stroke above, which locates x_1 on the horizontal axis, *i.e.*,

- vertical from diagonal to horizontal axis,

is followed by the first stroke of the second step,

- vertical from horizontal axis to graph,

which may be combined into a single stroke,

- vertical from diagonal to graph.

Thus the iterated sequence, beginning with x_0 on the horizontal axis, is:

- vertical from horizontal axis to graph,
- horizontal from graph to diagonal,
- vertical from diagonal to graph,

FIGURE 2-9.

The four strokes from a point on the horizontal axis to its image on the horizontal axis.

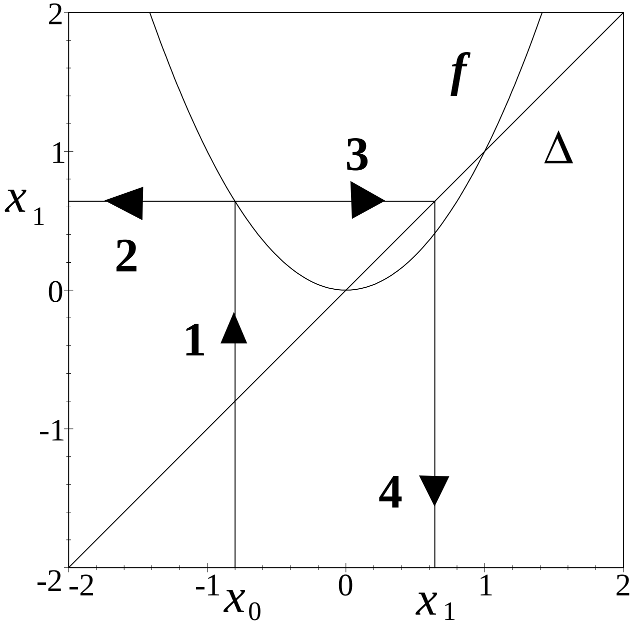


FIGURE 2-10.

The abbreviated three-stroke method.

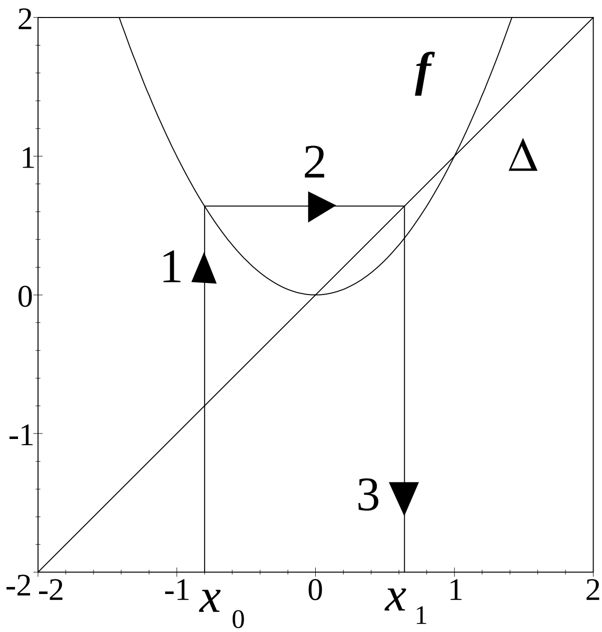


FIGURE 2-11.

One step, consisting of two strokes on the diagonal.

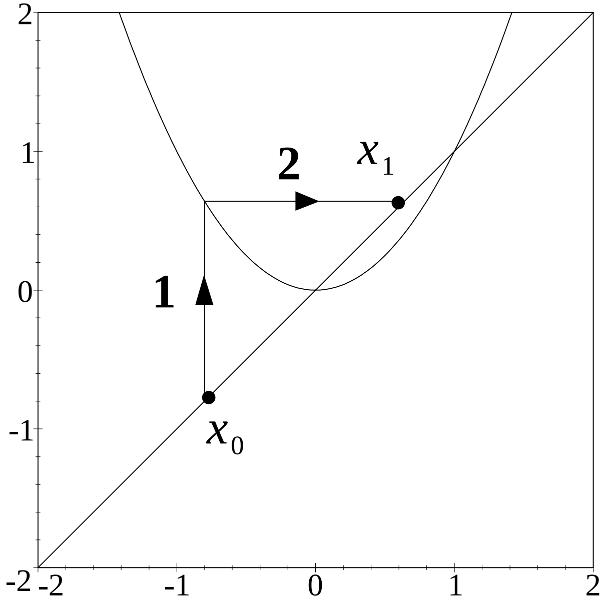
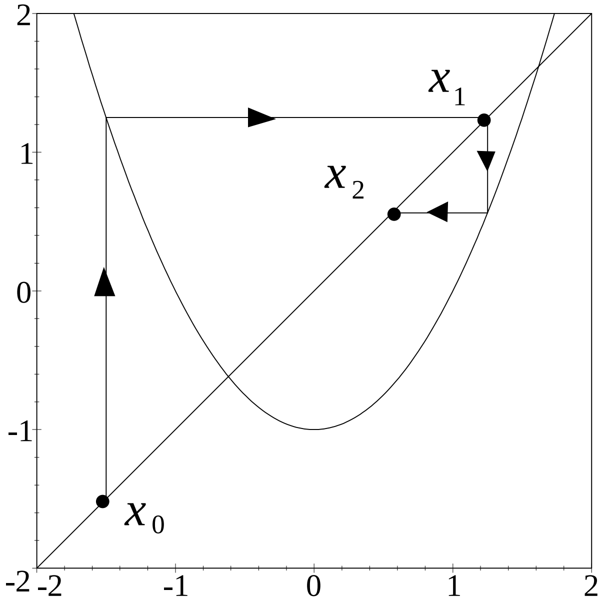


FIGURE 2-12.

The staircase method of Koenigs and Lemeray.



and continue. After the first step, we may pretend that we are jumping about on the diagonal, which after all is just another copy of the domain interval I. Each step has two strokes,

- vertical to the graph; horizontal to the diagonal

(which we may remember by the mnemonic, *vertigo-horrid*), as shown in Fig. 2-11. That is the graphical method of Koenigs-Lemeray, also known as the *staircase method*, or *cobweb construction*. Using it, we may quickly follow trajectories for several jumps on the diagonal. See Fig. 2-12.

Using this method, one may graphically verify this useful fact: if a point returns to its starting point after two iterations of the map, the starting point is either a fixed point or a 2-periodic point of the map.

2.4 ATTRACTORS, BASINS, AND BOUNDARIES

Try out the staircase method using the *Myrberg map*,[1] $f(x) = x^2 - c$, with the entire real line as the domain, and various values for the control parameter, c. You may quickly find that some trajectories converge to a *fixed point*, while others run off to positive infinity (upper right) on the diagonal. The fixed points are seen immediately as the crossing points of the graph and the diagonal, and are defined by the property: $f(x) = x$.

An interval is called *trapping* if it is mapped into itself, and *invariant* if it is mapped exactly onto itself. If a bounded interval is trapping, then all of its trajectories are trapped inside, and must converge to a closed, invariant, and bounded limit set. These limit sets are the *attractors* of the map. Attractors may be classified in three categories:

- a *point attractor* is a single point,
- a *cyclic attractor* is a finite set of points, and

1. Myrberg was one of the first to study the bifurcation sequence of this map. See the Bibliography for references to his work.

- a *chaotic attractor* is any other type of attractor.[1]

The *basin* of an attractor is the set of all points tending to that attractor. The domain is decomposed into the basins of different attractors, including the *basin of infinity*, which consists of all points whose trajectories run away from any bounded set.

The *boundaries* of the basins, also called *frontiers* or *separatrices*, are of primary importance in dynamical systems theory. A detailed study of a map results in a *portrait*, in which the domain is decomposed into basins, one attractor shown in each.

2.5 BIFURCATIONS

As in the Myrberg map, $f(x) = x^2 - c$, we frequently encounter maps which depend on a parameter. As the parameter is changed, the portrait of the attractive set of the map may change gradually and insignificantly; however, as certain special values of the parameter are crossed, there may be a sudden and significant change in the portrait of the map. These special values are called *bifurcation points*, and the sudden changes in the portrait are called *bifurcations*. At the present time, dynamical systems theory does not have a satisfactory and rigorous definition of bifurcation, but the subject is now evolving through the study of examples. In fact, the goal of this book is to describe some of these examples, in a two-dimensional context.

In the current one-dimensional context, we again have the benefit of an excellent visualization device, the *response diagram*. In the case of a single control parameter, this is a two-dimensional graphic in which the vertical axis represents the domain of the map and the horizontal axis represents the control parameter. Above each point on the horizontal axis, the portrait of the corresponding map is indicated, with its attractors, basins, and basin boundaries. For a one-parameter family of maps of a two-dimensional domain, the response diagram is three-dimensional, as we will soon see.

1. This reflects the fact that different definitions of chaos abound in the literature.

2.6 EXEMPLARY BIFURCATION

The simplest bifurcations are the *fold* and the *flip*. These may involve changes to any kind of attractor. To introduce the basic concepts of bifurcation theory, however, we will describe the fold bifurcation in the simplest case, which involves point attractors.

The fold bifurcation is a *catastrophic bifurcation*. This means that, as the control parameter varies, an attractor appears or disappears suddenly. In this event, as shown in Fig. 2-13 with the control parameter moving to the right on the horizontal axis, a fixed point appears, and immediately separates into a pair of distinct fixed points. One is an attractor, the other, a repellor. The repellor is shown below the attractor. Points between the two fixed points are attracted to the upper fixed point, and repelled by the lower fixed point. These tendencies are indicated by the arrows in Figures 2-15 to 2-17.

To understand the mechanism of this bifurcation, we now turn to a specific example, the Myrberg family of maps, $f(x) = x^2 - c$. The graph of a map of this family is an upward-opening parabola, with the vertex on the vertical axis at distance c below the horizontal axis. As c increases, the parabola moves downward. Three cases of this graph are shown in Figs. 2-14, 2-15, and 2-16.

In the first case, Fig. 2-14, with $c = -0.5$, the parabola does not meet the diagonal because for this value of c, there are no fixed points. All trajectories tend upward without bound, to infinity.

In the next case, Fig. 2-15, with $c = -0.25$, the parabola meets the diagonal in a single point, which is the fixed point $x = 0.5$, corresponding to this value of c, the bifurcation value. Trajectories approach from below, but depart from above.

In the last case, Fig. 2-16, with $c = 0$, the parabola cuts the diagonal in two points, the fixed points $x = 0$ and $x = 1$, which are, respectively, an attractor and a repellor.

FIGURE 2-13.

The response diagram of a fold bifurcation. The state space, where the dynamics occur, is vertical. The control parameter, *c*, is horizontal. The parabolicoid curves locate the fixed points of the maps. The bifurcation occurs, in the Myrberg example, when $c = -0.25$.

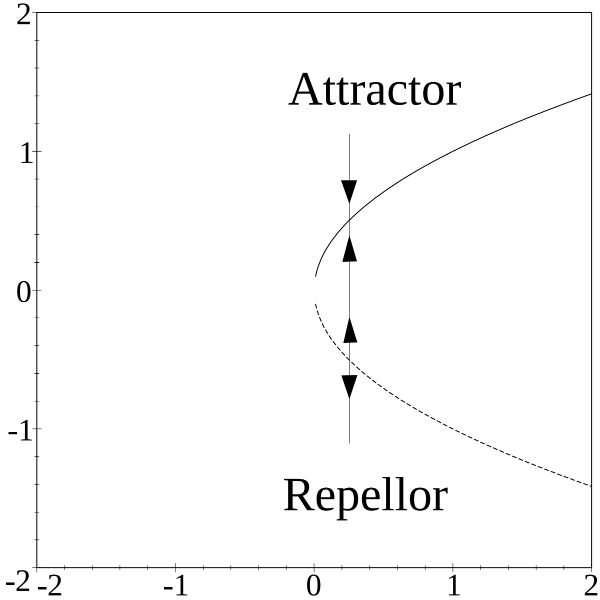


FIGURE 2-14.

A typical member of the Myrberg family, before the fold bifurcation. The graph of the map is entirely above the diagonal, so there are no fixed points.

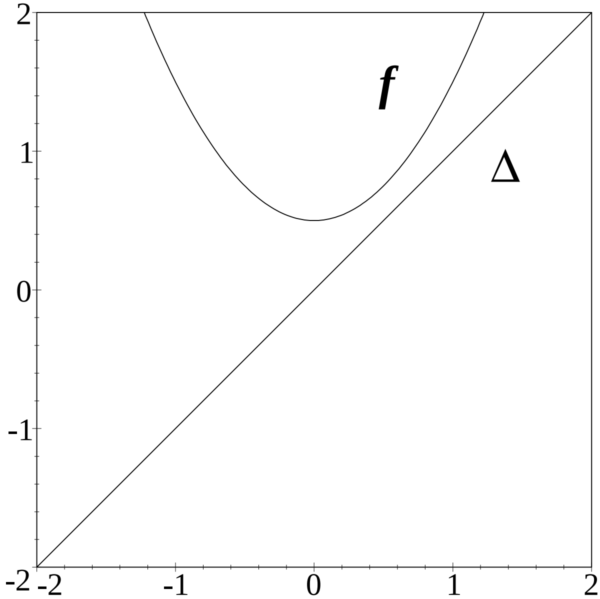


FIGURE 2-15.

At the fold bifurcation. The graph of the map has made contact with the diagonal at a single fixed point.

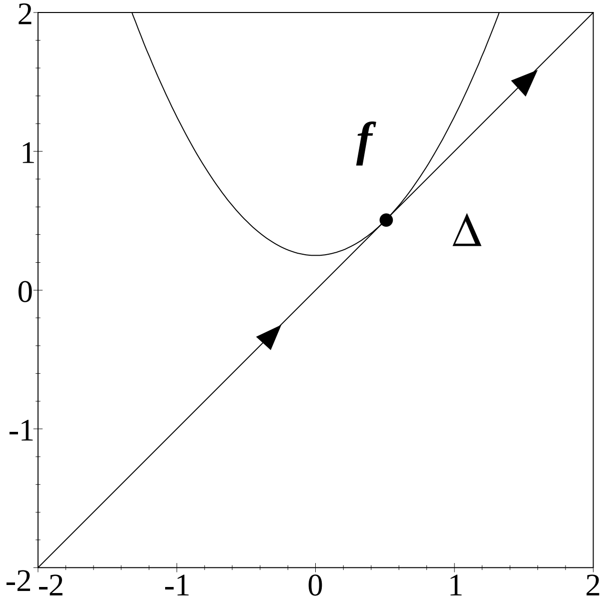


FIGURE 2-16.

After the fold bifurcation. The graph now meets the diagonal in two points, both fixed.

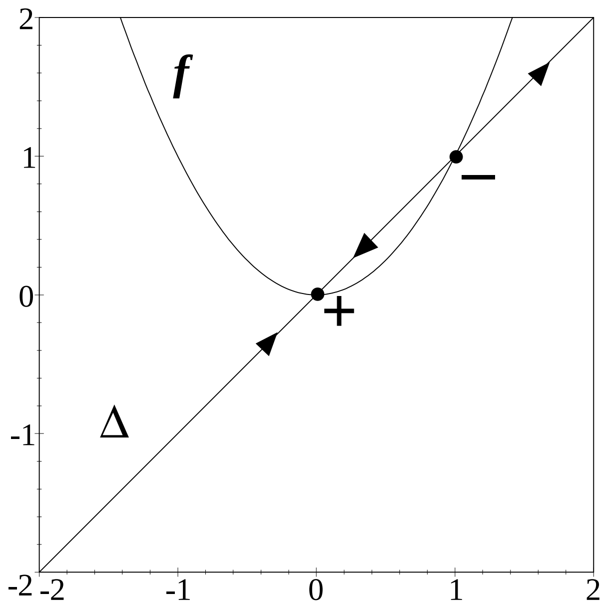
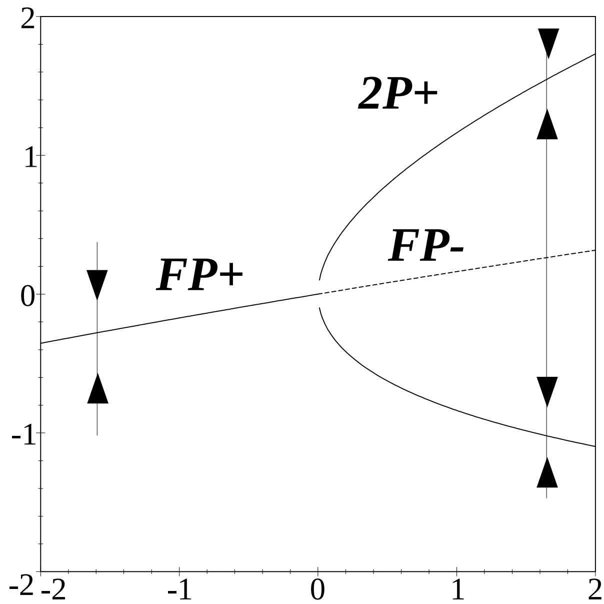


FIGURE 2-17.

The response diagram of the flip bifurcation.



The flip is a *subtle bifurcation*. This means that, in contrast to catastrophic and explosive bifurcations, its effect is too subtle to observe at the moment of bifurcation when the control parameter passes its critical value, but becomes apparent later, as the parameter continues to increase. In the flip, a point attractor loses its attractiveness. From it is emitted a cyclic attractor of period 2.

A response diagram of this event is shown in Fig. 2-17. To the left of the bifurcation value of the control parameter (horizontal axis) there is a single fixed point, and it is an attractor, *FP+*. The attraction in the vertical state space is shown by the heavy arrows. To the right, there is still only one fixed point, but it is a repellor, *FP-*. But there is also a 2-cycle, which is attractive, *2P+*. Looking only at the attractors in the picture, we see that the attractive point has been replaced by an attractive 2-cycle, as the control parameter moves to the right. At first, the two points of this 2-cycle are very close together, then they gradually separate.

We now move on to two dimensions.