CHAPTER 3

BASIC CONCEPTS IN 2D

The basic concepts named in the Introduction, and described in the preceding chapter in a 1D context, apply with little modification in the 2D context which is our main concern in this book. We no longer have the convenience of a visible graph of the map, however, because the graph of a 2D map is a 2D surface in a 4D space. Therefore, we must be satisfied with a frontal view of the 2D domain of the map, in which we try to visualize as much as possible.

3.1 MAPS

As before, by *map* we mean a continuous function from the *domain* to itself, $f: D \to D$. From now on, the domain will be a two-dimensional subset, usually an open subspace, of the plane. For example, D might be an open rectangle (that is, not containing its boundary) or the whole plane. The *images* and *preimages* of a point, the *one-to-one* property, and *noninvertibility* are defined as in 2.1. We now consider noninvertible maps, in 2D.

Note: The complex number maps familiar from the fractal theories of Fatou, Julia, Mandelbrot, and others may be regarded as real 2D maps. Thus, they fit in the context of this book.

3.2 MULTIPLICITIES AND CRITICAL CURVES

In the context of a given map, we define the *layer set,* Z_n, as the set of points having exactly n preimages of rank 1, where n is a natural number: 0, 1, 2,..., and so on. Those layer sets that are nonempty open sets are the *multiplicity zones*. Points on the boundaries of the zones, generally, are *critical points* of the map. In general, these sets will not exhaust the domain, as there may be points that have infinitely many preimages of rank 1, and thus do not belong to Z_n for any n. Even polynomial maps may have this problem, but generic (that is, almost all) polynomial maps have the following nice properties:

- there are only a finite number of layer sets;
- they exhaust the domain;
- the zones of multiplicity fill almost all of the domain;
- all layer sets that are not multiplicity zones consist of critical points, arranged in a set of piecewise smooth curves.

A map having these nice properties is called a *finitely folded map*, and a curve consisting of critical points is called a *critical curve of rank 1*, and is denoted by L.[1] The image of a critical curve of rank 1 is a critical curve of rank 2, denoted L_1, and so on.

Note: For a given map T, $L = T[L_{-1}]$, where L_{-1} is the *critical curve of rank 0*, and may be thought of as the set of "coincident preimages" of points of L.

3.3 AN EXAMPLE

For example, let D be the entire plane. The polynomial map $f: D \rightarrow D$ defined by $f(x, y) = (u, v)$, where,

$$u = ax + y$$
$$v = b + x^2 \qquad \textbf{(EQ 3)}$$

is finitely folded. We will set $a = -0.7$, and $b = 1.0$.

1. In the original literature, L is usually denoted by LC, for the French term, *ligne critique*. More rigorous definitions may be found in Appendix 3.

Figure 3-1 shows part of the domain of the map, the (*x*, *y*) space *D*, containing the critical curve of rank 0, L_{-1}, which coincides with the *y* axis. It divides the domain into two regions, denoted R_1 and R_2. Figure 3-2 shows part of the image of the map *f*, in the (*u*, *v*) space, with the zones of multiplicities zero and two, Z_0 and Z_2, separated by the critical curve, *L*. Figure 3-3 shows the two spaces, superimposed.

There is a *folding* of the (*x*, *y*) space, on the critical curve of rank 0, L_{-1}, followed by a nonlinear deformation, a rotation, and a movement to the right, into the space of (*u*, *v*). The entire (*x*, *y*) space ends up on the zone Z_2, with the critical curve L_{-1} moving onto the critical curve, *L*. We may visualize the two regions of the (*x*, *y*) space, R_1 and R_2 folded onto one another, then distorted and pressed down onto Z_2. Actually, the two regions mapped are onto one.

The motion, visualized in this way, may be reversed. This provides a method to visualize the action of the inverse mapping as well. A small area in the (*u*, *v*) space on the right, if contained entirely within Z_2, will unfold into two small regions in the (*x*, *y*) space on the left, one in the region R_1, the other in R_2.

As we wish to iterate the map, and to visualize the trajectories, attractors, and basins, of our discrete dynamical system, it will be useful (although initially confusing) to superimpose the (*u*, *v*) space on top of the (*x*, *y*) space. Then, as a weak substitute for the graphical method of Koenigs-Lemaray in the 1D case, we apply the motion from Fig. 3-1 to Fig. 3-2 again and again. The domain, *D*, is mapped into itself repeatedly. The curve L_{-1} moves onto the curve *L*, which in turn moves onto the curve L_1, and so on. This superimposition is shown in Fig. 3-3. This portrait is the basis of the *method of critical curves*, which is the main method of this book.

FIGURE 3-1.

The domain of (x, y) divided by the critical curve, $x = 0$.

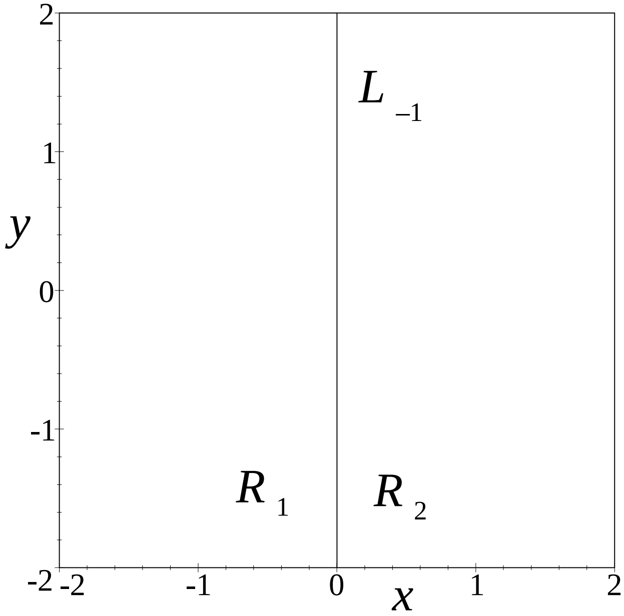


FIGURE 3-2.

The range of (u, v) divided by the critical curve, $v = b = 1.0$. The image of the map is above L.

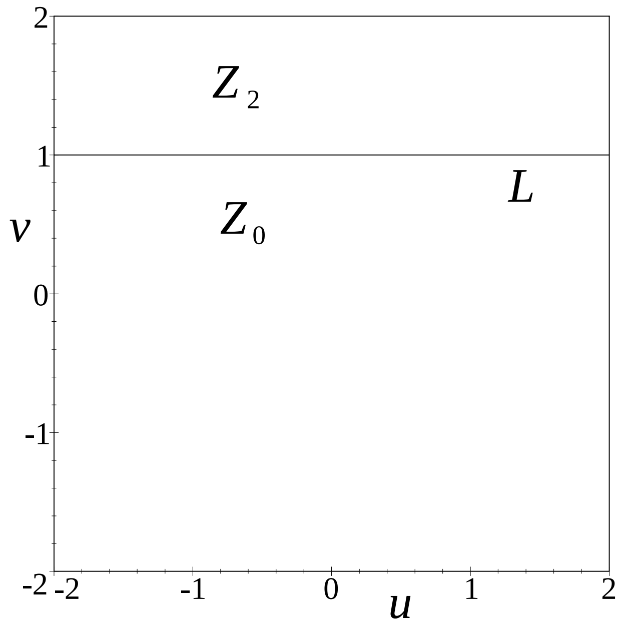
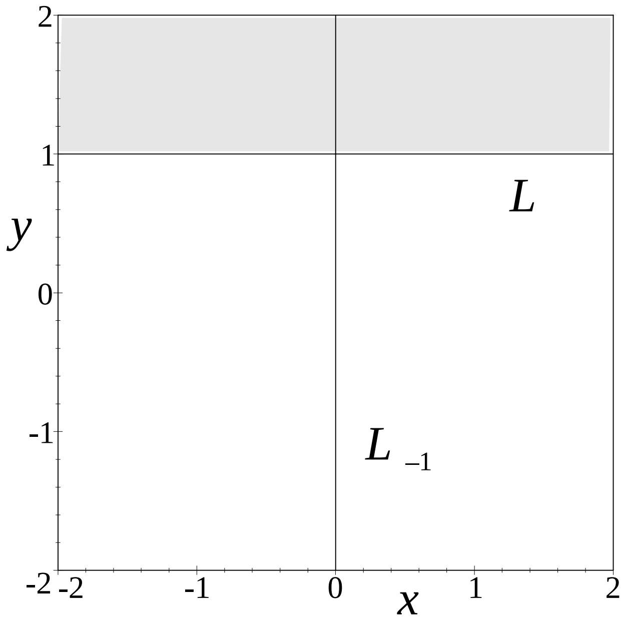


FIGURE 3-3.

The image of the map, superimposed on the domain.



3.4 TRAJECTORIES AND ORBITS

In the 2D context, *trajectories* and *orbits* are defined exactly as in 2.3,. but here they must be plotted in the two-dimensional domain. This is particularly appropriate for computer-generated plots. And in this computational method, the *attractors* and their *basins* may be discovered by experiment. We usually find the critical curve of rank – 1 manually by the standard method of vector calculus (involving the vanishing of the Jacobian determinant, see Appendix 3), then enter its symbolic description into the computer program, which can then plot the higher-order iterates. The method of critical curves is based on experiments such as this.

As in the 1D case, there are special kinds of orbits which are important qualitative features of the dynamics of an iterated map. First among these are the *fixed points*, which are unmoved by the map. The different types of fixed points are defined by the motions of nearby points. The classification is based on the differential calculus and linear algebra of two-dimensional spaces, but we will give here only the results. Excepting certain unusual cases, there are five kinds of so-called *generic* fixed points in this classification. The five types are illustrated in Figs. 3-4 to 3-8.

Another special type of orbit of great importance in the qualitative theory is the *periodic orbit*, or *cyclic orbit*, or *cycle*, which consists of a finite set of points. The map permutes the points of the orbit cyclically: If there are *n* points in the orbit, each of the points returns to its original position after exactly *n* iterations of the map. The number *n* is called the *period* of the orbit, which is also called an *n-cycle*. A point of an *n*-cycle is said to have *prime period n*, and is also a fixed point of the map iterated *n* times. Periodic points are classified according to their type as a fixed point of the iterated map.

FIGURE 3-4.

Attractive focus. All nearby points are attracted and spiral toward the fixed point.

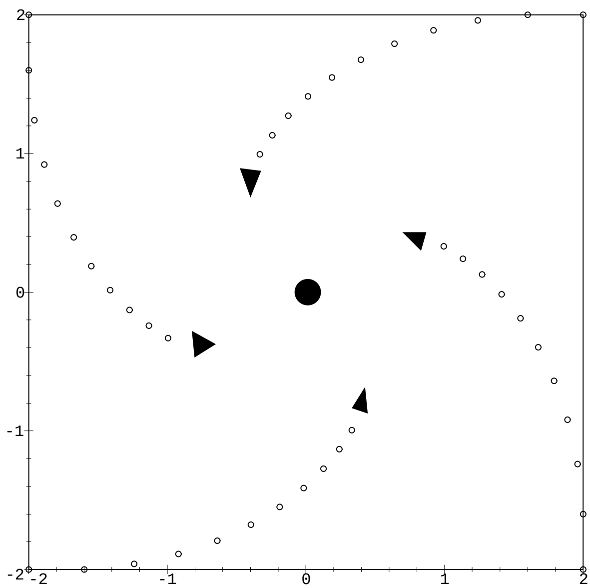


FIGURE 3-5.

Attractive node. All nearby points are attracted, and tend to approach along a curve through the fixed point.

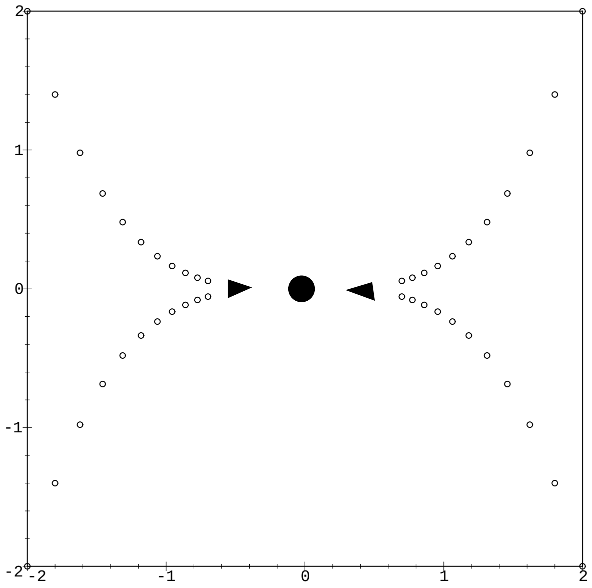


FIGURE 3-6.

Saddle. A repellor, most nearby points are attracted, and then repelled along a curve through the fixed point.

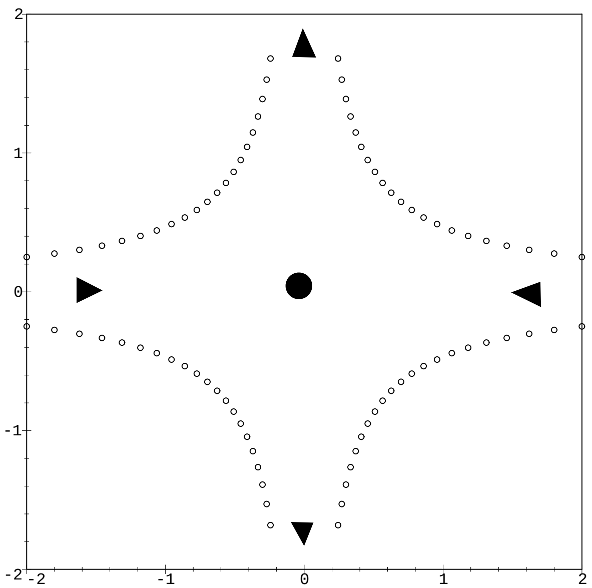


FIGURE 3-7.

Repelling node. All nearby points are repelled, and tend to depart (at least briefly) along a curve through the fixed point. The opposite of an attractive node.

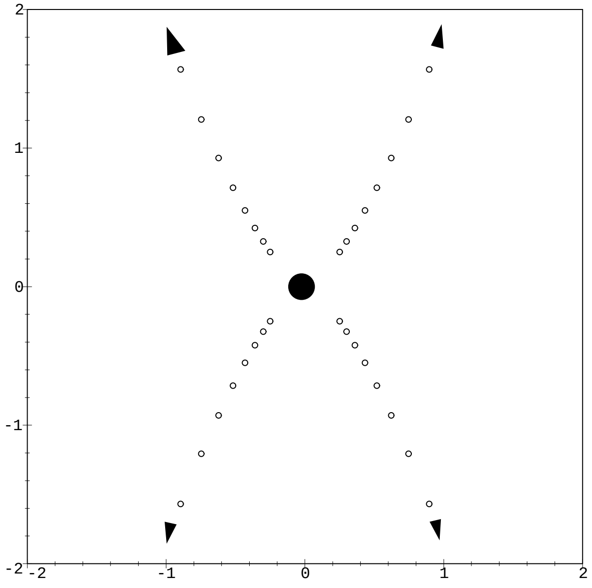
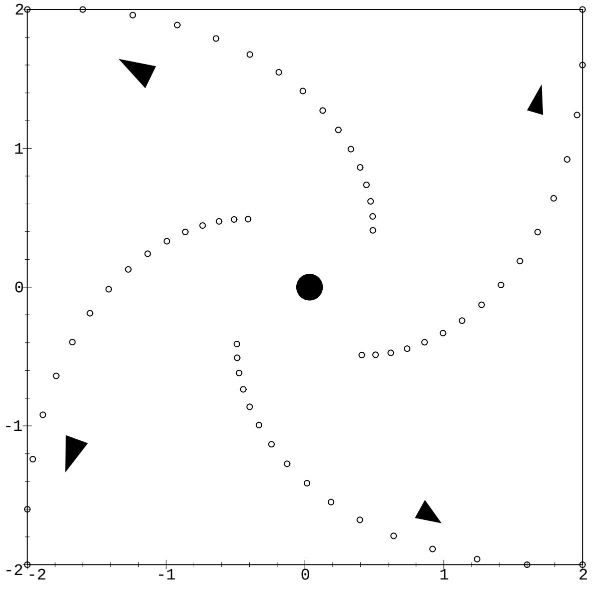


FIGURE 3-8.

Repelling focus. All nearby points depart, spiralling away from the fixed point. The opposite of an attractive node.



3.5 ATTRACTORS

As in the 1D case, there are three types of *attractors*:

- *static attractors*, also called *attractive fixed points*;
- *periodic attractors*, also called *cyclic attractors*; and
- *chaotic attractors*.

The static attractors are fixed points which are attractive, that is, the trajectories of all nearby points are attracted to them. Of the five types of fixed points illustrated in Figs. 3-4 to 3-8, two are attractive, and three repelling. Periodic attractors are periodic orbits (orbits of trajectories that cycle around a finite point set) which are attractive.

Chaotic attractors are more complicated sets which are attractive. For the mathematically inclined, technical definitions are given in Appendix 2. For others, these concepts will gain meaning through examples later in the book.

3.6 BIFURCATIONS

The informal definition of *bifurcation*, in 2D, is the same as in 1D. Again, there are subtle and catastrophic bifurcations, and other distinctions such as local versus global bifurcations. These are best understood in the examples dissected in detail in the following chapters.

In the 2D context, a one-parameter family of maps may be displayed in a *response diagram*, in which the bifurcations may be seen and analyzed. This is a 3D plot in which the domain of the maps, a 2D set, is arrayed vertically, and moved along a horizontal axis representing the control parameter. In each of these vertical planes, the portrait of attractors, basins, and boundaries must be visualized. In practice, this is a challenging task of computer graphics, and we usually seek a simpler display. The technique we adopt for this book, which is well accommodated by computer graphic animation technology and CD-ROM media, is the animated movie. Thus, we translate the control parameter into the time dimension and view the domain of the map head on, watching the attractor-basin portrait adjust itself to a time-changing control parameter.

The method of characteristic curves becomes a strategy for the analysis of these bifurcation movies. So, on to the exemplary bifurcation sequences.